

THE PROPERTIES OF THE  
GROUPS  $O_N$ ,  $SO_N$ ,  $S_n$  AND  $A_n$

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by

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To my wife  
To my children

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## PREFACE

The problems of group theory applied in physics often are reduced to the calculation of the dimensions, the branching rules, the resolution of the Kronecker products, the symmetrized powers and the classification of the irreducible representations (irreps) for a wide variety of groups. The orthogonal  $O_N$  and its subgroups, especially the rotation groups  $SO_N$ , symmetric groups  $S_n$  and the alternating groups  $A_n$  have been of special interest to physicists.

The  $n$ -dimensional rotation groups play an important role in many areas of physics and chemistry. They arise, for example, in the description of symmetrized orbitals in quantum chemistry (Wybourne 1973), in fermion many body theory (Fukutome et al 1977), in boson models of nuclei (Arima and Iachello 1976), grand unified theories (Gell-Mann et al 1978) and in supergravity theories (Cremmer and Julia 1978).

Interest in the rotation groups has greatly increased in recent times with study of candidate groups for grand unified theories of the weak, electromagnetic and strong interactions. The group  $SO_{10}$  appears to be of particular significance (Fritzsch and Minkowski 1975, Chanowitz et al 1979, Buras et al 1978, Georgi and Nanopoulos, 1979, Witten 1979).

The symmetric group has long been of interest to physicists and chemists who have sought to exploit the permutational symmetry associated with many fermion and many-boson systems.



For  $A_n$  in physics the isomorphisms  $C_3 \sim A_3$ ,  $T \sim A_4$  and  $I \sim A_5$  are well-known in solid state and molecular physics (Lax 1974).

The subject of my thesis is devoted to problems concerning dimensions, branching rules and the resolution of the Kronecker products etc for the groups  $O_N$ ,  $SO_N$ ,  $S_n$  and  $A_n$ .

This thesis is organized as follows: In chapter 1 a brief statement is given about the results of the theory of representations of group  $S_n$  both in ordinary and spin representations that will later be used. On spin representations, I prefer the concept of projective representations to that of the ordinary representation of double groups. Given a brief statement about projective representation I restate the familiar results gotten from double groups from the standpoint of projective representations. In the end of this chapter I present some results relevant to associated and self-associated representations.

In chapter 2 I provide some review material on the symmetric functions including basic symmetric functions, Schur functions (S-functions), raising operators etc. In Sec. 2.7 we use the concepts of partitions, frames and numberings to give a simple method to evaluate the outer product of S-functions and the skew S-functions. While in Sec. 2.8 in addition to the known S-function series such as  $A, B$  we also give some new series (e.g.  $R, S \dots$ ) and their relations which will be used in the representations of  $SO_N$ .

Chapter 3 is devoted to the relation between the symmetric functions and the representations of groups. The relationship between S-functions, Q-functions and the representation theory of groups is outlined. I present the definitions of the inner product of S-functions and Q-functions which play an important role for resolving the Kronecker product of the spin and ordinary irreps of  $S_n$  and point out the relation between branching rules, skew S-functions and Q-functions.

The results given in chapter 4 are explicit formulae for a complete set of fundamental products from which all possible products of irreps of  $O_N$  and  $SO_N$  may be evaluated both for  $n = 2v$  and for  $n = 2v + 1$ . The explicit resolution of the basic Kronecker squares into their symmetric and antisymmetric parts is then given, followed by a complete resolution of the Kronecker cubes of the basic spin irreps of  $SO_{2v+1}$  and  $SO_{2v}$ , together with a prescription for analysing explicitly the Kronecker fourth powers of these irreps. These results permit analysis of the Kronecker second, third and fourth powers of any irreps (spin or tensor) of the groups  $SO_{2v+1}$  and  $SO_{2v}$  to be made unambiguously. These results are given in a general form that is essentially independent of  $N$ , the dimension of  $SO_N$ .

The problem of resolving the Kronecker products of ordinary representations of  $S_n$  has received considerable attention, and techniques have been developed that obviate the need to use explicit character tables (Murnaghan 1937, 1938, Littlewood 1958a,b, Butler and King 1973). Furthermore

many of the results have been given in an  $n$ -independent form using a "reduced notation" for labelling the irreducible representations of  $S_n$ . But the spin (or projective) irreps of  $S_n$  have received far less attention. As long ago as 1911, Issia Schur, having previously investigated the representations of any finite group by linear fractional (Schur 1904, 1907) directed his attention to the study of spin representation of  $S_n$  (Schur 1911). Methods of constructing spin characters tables of  $S_n$  are of recent origin (Morris 1962a, Read 1977). Remarkably little is known about the resolution of Kronecker products involving the spin representations apart from the explicit use of character tables. This contrasts strongly with the corresponding reduction of the ordinary representation of  $S_n$ . In chapter 5 I give attention to the problems mentioned above. I establish an  $O_n \supset S_n$  embedding and the formation of branching rules for  $O_n \downarrow S_n$  is then considered, leading to a reduced notation for the spin representations of  $S_n$ , making possible many  $n$ -independent results. The first application is to discuss the  $n$ -independence of the dimensions of the spin representations of  $S_n$ . In order to facilitate the reduced notation, a special Young raising operator  $R_{0j}$  is introduced. These results, together with consideration of difference characters of  $S_n$ , give a general procedure for resolving arbitrary Kronecker products without the explicit use of character tables. We are then able to use the method of plethysm to resolve Kronecker squares of the spin representations into their symmetric and antisymmetric parts, and eventually to classify the spin irreps as to their

orthogonal, symplectic or complex characters.

In the last chapter, chapter 6, I extend the reduced notation developed in chapter 5 to  $A_n$ , the subgroup of  $S_n$ , leading to an essentially  $n$ -independent treatment of the properties of the representations of  $A_n$ . Branching rules for  $S_n \downarrow A_n$  are developed. The difference character for the irreducible representation of  $A_n$  are established and used to establish a series of algorithms for evaluating Kronecker products and plethysms of spin and ordinary irreps of  $A_n$ . In the concluding section the systematic classification of the irreps of  $A_n$  is given.

I would like to express my gratitude to my supervisor, Professor B.G. Wybourne, for his continued support and guidance, particularly for demonstrating the power of symmetric functions and difference characters to me. Also I would like to express thanks to Dr P.H. Butler for numerous discussions. Appreciation is also recorded of Dr R.C. King in preparing the manuscript of the material included in chapter 4 for publication. Lastly, I would like to thank my wife, Wang Shuxian, for looking after my family and children when I am on leave and Janet Warburton for her excellent job of typing a very difficult manuscript.

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## CHAPTER 1

### THE REPRESENTATIONS OF THE SYMMETRIC GROUPS

#### 1.1 THE GROUP $S_n$

The  $n!$  permutations of  $n$  objects form a group called the symmetric group  $S_n$ .

Any permutation can be resolved into the product of cycles. If  $v_1$  = no. of one-cycles,  $v_2$  = no. of two-cycles, etc., then since there are just  $n$  symbols for a given permutation in  $S_n$

$$1v_1 + 2v_2 + \cdots + nv_n = n \quad (1.1)$$

The cycle structure may be designated as  $(1^{v_1} 2^{v_2} \cdots n^{v_n})$  or just  $(v)$ . All permutations in  $S_n$  which have the same cycle structure  $(v)$  form a conjugate class in  $S_n$ . Thus each solution of (1.1) for positive integers  $v_1, v_2, \cdots, v_n$  gives a class in  $S_n$ ; hence the number of classes is just the number of such solutions. Let

$$\begin{aligned} v_1 + v_2 + \cdots + v_n &= \lambda_1 \\ v_2 + \cdots + v_n &= \lambda_2 \\ &\cdot \quad \cdot \\ &\cdot \quad \cdot \\ &\cdot \quad \cdot \\ v_n &= \lambda_n \end{aligned} \quad (1.2)$$

then

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = n = w_\lambda \quad (1.3a)$$

and

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq 0 \quad (1.3b)$$

Thus each solution of (1.1) corresponds to an ordered partition of  $n$  and  $w_\lambda$  is called the weight of partition  $(\lambda)$ .

Ordinarily a partition  $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_{p_\lambda}, 0, 0, 0)$  of  $n$  is written  $(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_{p_\lambda})$  i.e. we omit the  $\lambda_i$  that are zero. Also, if several of the  $\lambda_i$  are equal we use exponents to shorten the notation. Thus the partitions (4000), (3100), (2200), (2110) and (1111) of 4 are usually written in the abbreviated forms

$$(4), (31), (2^2), (21^2), (1^4)$$

The number of classes in  $S_n$  is equal to the number of partitions into positive integers such that (1.3) is satisfied.

Given a partition as in (1.3) there is a corresponding cycle structure, namely

$$\begin{aligned} v_1 &= \lambda_1 - \lambda_2 \\ v_2 &= \lambda_2 - \lambda_3 \\ &\cdot \quad \cdot \\ &\cdot \quad \cdot \\ &\cdot \quad \cdot \\ v_n &= \lambda_n \end{aligned} \quad (1.4)$$

Cycle structures involving an even number of even length cycles correspond to even permutations and are called even classes while all other cycle structures are odd and are referred to as odd classes.

For later convenience we shall adopt the convention of listing the cycle structures in order of their decreasing length and omit all cycles with exponents  $v_i = 0$ . Thus in  $S_4$ , we designate the classes as  $(1^4), (21^2), (2^2), (31)$  and  $(4)$  with the  $(1^4), (2^2)$  and  $(31)$  classes involving even permutations only.

The number of distinct permutations of  $S_n$  having the cycle structure  $(n^{v_n} \dots 2^{v_2} 1^{v_1})$  is

$$g(v) = n!/v_1! 1^{v_1} v_2! 2^{v_2} \dots v_n! n^{v_n} \quad (1.5)$$

Thus for  $S_4$ , we have

partition	cycle structure	no. of element in class
(4)	$(1^4)$	1
(31)	$(21^2)$	6
$(2^2)$	$(2^2)$	3
$(21^2)$	$(31)$	8
$(1^4)$	$(4)$	6

We note that the total number of elements is equal to  $4!$  which is the order of  $S_4$ .

## 1.2 THE PROJECTIVE REPRESENTATIONS OF FINITE GROUPS

We now consider a few of the relevant properties of irreducible projective representations (i.p.r) of finite groups (Schur 1911, Curtis and Reiner, 1962 and Dornhoff, 1971).

Let  $G$  be a finite group,  $k$  a field and  $k^X$  the multiplicative group of  $k$ ,  $V$  a finite dimensional  $k$ -vector space. A projective representation of  $G$  on  $V$  is a mapping  $T: G \rightarrow GL(V)$  such that for all  $x, y \in G$

$$\begin{aligned} T(x) T(y) &= \alpha(x, y) T(xy) \\ T(e) &= I \end{aligned} \tag{1.6}$$

where  $\alpha(x, y) \in k^X$ . The function  $\alpha: G \times G \rightarrow k^X$  is called a factor set of  $T$  and  $T$  an i.p.r. if  $V$  has no proper subspace invariant under  $T(x)$ ,  $x \in G$ . Furthermore

$$\begin{aligned} \alpha(x, y) \alpha(xy, z) &= \alpha(x, yz) \alpha(y, z) & x, y, z \in G \\ \alpha(e, e) &= 1 & x \in G \end{aligned} \tag{1.7}$$

If  $\alpha = 1$  then  $T$  is an ordinary representation. Two factor sets  $\alpha$  and  $\beta$  of  $G$  are termed equivalent if there is a function  $\gamma: G \rightarrow k^X$  such that for all  $x, y$ ,

$$\alpha(x, y) = \beta(x, y) \gamma(x) \gamma(y) \gamma^{-1}(xy) \tag{1.8}$$

The set  $H^2(G, k^X)$  of all equivalence classes under (1.8) with multiplication  $\{\alpha\}\{\beta\} = \{\alpha\beta\}$  well defined, forms an



Abelian group of equivalence classes of the factor sets and is known as the Schur multiplier of  $G$  over  $k$ . For  $S_n$  we have (Davies and Morris, 1974).

$$H^2(S_n, C^X) = C_2 = \{r\} \quad (n \geq 4) \quad (1.9)$$

if  $r = 1$  the  $T$  of  $S_n$  will be called an ordinary irrep. While if  $r = -1$   $T$  will be called an projective or spin irrep of  $S_n$ .

The centralizer  $C(x)$  of an element  $x \in G$  is the collection of elements  $s \in G$  such that  $s x s^{-1} = x$ . If  $\alpha$  is factor set of  $G$ , an element  $x \in G$  will be termed an  $\alpha$ -regular element if

$$\alpha(x, s) = \alpha(s, x) \quad (1.10)$$

for all  $s$  in the centralizer of  $x$  in  $G$ . If  $x$  is  $\alpha$ -regular then every element which is conjugate to  $x$  in  $G$  is  $\alpha$ -regular and hence we may speak of an  $\alpha$ -regular class.

Two major differences between projective and ordinary irreps must be noted. The first one concerns the character  $\chi^T$ . For ordinary irreps, as is well known,  $\chi^T$  are the class functions, whereas in projective irreps we have (Altmann, 1979)

$$\chi^T(g g_i g^{-1}) = \alpha(g, g_i g^{-1})^{-1} \alpha(g_i, g^{-1})^{-1} \alpha(g, g^{-1}) \chi^T(g_i) \quad (1.11)$$

The sufficient condition for the character to be a class function over the whole group  $G$  is that equation

$$\alpha(gg_i g^{-1}, g) = \alpha(g, g_i) , \quad \forall g \in G \quad (1.12)$$

obtains for all regular elements  $g_i \in G$ . This class function vanishes over all the irregular elements of  $G$ .

The second difference is that the number of inequivalent irreps that can be constructed for a given factor system is not, as for ordinary irreps, equal to the number of classes. The number of distinct inequivalent i.p.r. of  $G$  with the factor set  $\alpha$  is equal to the number of  $\alpha$ -regular classes of  $G$  and

$$\sum_i n_i^2 = g \quad (1.13)$$

where  $n_i$  are the dimensions of the inequivalent i.p.r's and  $g$  is the order of  $G$ .

For  $S_n$  the  $\alpha$ -regular classes fall into two categories:

(1) even permutation-classes containing only cycles of odd order; (2) odd permutation classes containing cycles of unequal orders. Thus for  $S_7$  we have the eight  $\alpha$ -regular classes

$(1^7), (31^4), (51^2), (3^21), (7)$	even
$(61), (52), (43)$	odd

while for  $S_8$  we have nine  $\alpha$ -regular classes

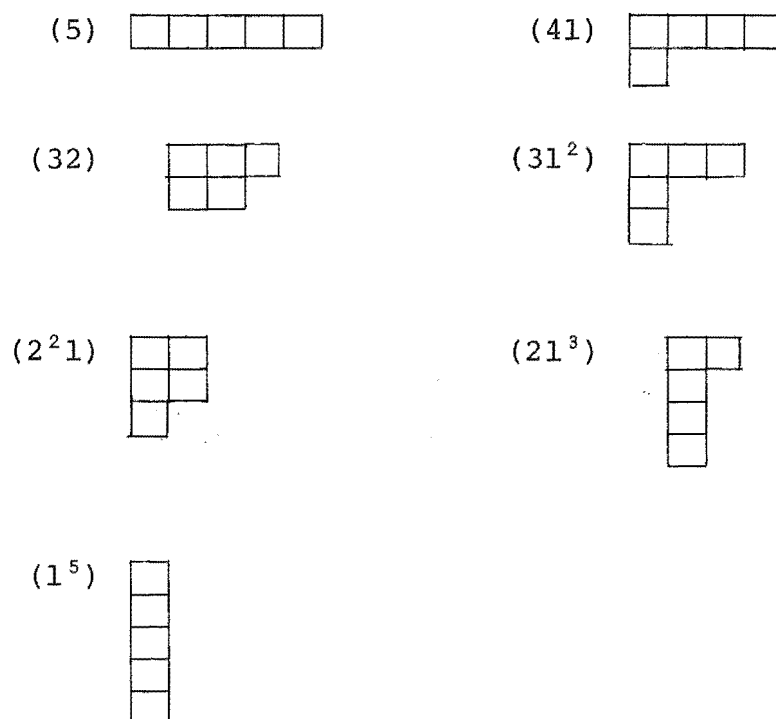
$(1^8), (31^5), (51^3), (3^21^2), (71), (53)$	even
$(8), (521), (431)$	odd

### 1.3 THE ORDINARY REPRESENTATIONS OF $S_n$

It is well known that for a finite group  $G$  the number of nonequivalent irreps of  $G$  is equal to the number of conjugate classes in  $G$ . For  $S_n$ , as mentioned in sec. 1.1, the number of conjugate classes is equal to the number of partitions of  $n$ , so we can use the partition  $(\lambda)$  to label the irreps of  $S_n$  designated by  $[\lambda]$ .

Every partition  $(\lambda)$  of  $n$  may be given a unique graphical representation in terms of  $n$  cells or nodes arranged in  $P_\lambda$  left-adjusted rows with the  $i$ -th row containing  $\lambda_i$  cells. These graphs are known as Young diagrams.

For  $n = 5$  we have the diagrams



For a given diagram  $(\lambda)$  we can obtain another diagram from  $(\lambda)$  by interchanging rows and columns called the associated partition of  $(\lambda)$  denoted by  $(\tilde{\lambda})$ , for example

$$(32) \quad \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \quad \widetilde{(32)} \quad \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

$$\therefore \widetilde{(32)} = (2^2 1)$$

If  $(\widetilde{\lambda}) = (\lambda)$  then partition is said to be self-associated.

For example the partition  $(31^2)$  of 5 is self associated.

If  $(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_{p_\lambda})$  then

$$(\widetilde{\lambda}) = (p_\lambda^{\lambda_1}, (p_\lambda - 1)^{\lambda_2}, \dots, 1^{\lambda_1 - \lambda_2}) \quad (1.14)$$

hence

$$\widetilde{(5421^3)} = (632^2 1)$$

It is well known from number theory (Hardy and Wright, 1954) that the number of partitions of  $n$  into odd and unequal parts is equal to the number of its self-associated partitions. Every self-associated partition  $(\lambda) = (\lambda_1, \lambda_2, \dots)$  can be related to a unique partition of odd and unequal parts  $(p) = (p_1, p_2, \dots, p_k)$  where

$$p_i = 2\lambda_i - 2i + 1 \quad (1.15)$$

we say  $(p)$  belongs to the self-associated partition  $(\lambda)$ .

The number of self-associated partitions for given  $n$  is readily seen to be

$$\sum B_m[(n-m^2)/2] \quad (1.16)$$

where  $B_m[q]$  is the number of partitions of  $q$  into at most

m parts, the summation is over  $m = 2, 4, \dots$  for n even and  $m = 1, 3, \dots$  for n odd and  $B_m(0)$  are always defined as

1. Table 1 gives the value of  $B_m[q]$  ( $m = 10, q \leq 10, q \leq 30$ ).

For example for  $n = 31$  the number of self-associated is

$$B_1[15] + B_3[11] + B_5[2]$$

$$= 1 + 16 + 2 = 19$$

and for  $n = 36$  is

$$B_2[16] + B_4[10] + B_6[0]$$

$$= 9 + 23 + 1 = 33$$

Another method of describing partitions, known as the Frobenius notations, will occasionally be used. The diagonal cells in a Young diagram beginning at the top left hand corner is called the leading diagonal. The number of cells on the leading diagonal is termed the rank of the partition. If  $r$  is the rank of the partition and there are  $a_i$  cells to right of the leading diagonal in the  $i$ th row and  $b_i$  cells below the leading diagonal in the  $i$ th column then partitions may be designated after Frobenius as

$$(\lambda) = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \quad (1.17a)$$

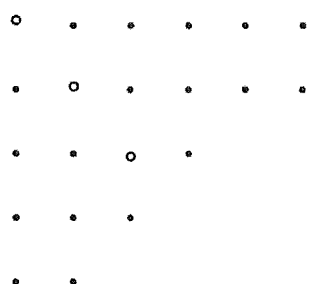
with

$$(\tilde{\lambda}) = \begin{pmatrix} b_1 & b_2 & \dots & b_r \\ a_1 & b_2 & \dots & a_r \end{pmatrix} \quad (1.17b)$$

TABLE 1: The value of  $B_m[q]$ 

$\begin{matrix} m \\ q \end{matrix}$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3	3
4	1	3	4	5	5	5	5	5	5	5
5	1	3	5	6	7	7	7	7	7	7
6	1	4	7	9	10	11	11	11	11	11
7	1	4	8	11	13	14	15	15	15	15
8	1	5	10	15	18	20	21	22	22	22
9	1	5	12	18	23	26	28	29	30	30
10	1	6	14	23	30	35	38	40	41	42
11	1	6	16	27	37	44	49	52	54	55
12	1	7	19	34	47	58	65	70	73	75
13	1	7	21	39	57	71	82	89	94	97
14	1	8	24	47	70	90	105	116	123	128
15	1	8	27	54	84	110	131	146	157	164
16	1	9	30	64	101	136	164	186	201	212
17	1	9	33	72	119	163	201	230	252	267
18	1	10	37	84	141	199	248	288	318	340
19	1	10	40	94	164	235	300	352	393	423
20	1	11	44	108	192	282	364	434	488	530
21	1	11	48	120	221	331	436	525	598	653
22	1	12	52	136	255	391	522	638	732	807
23	1	12	56	150	291	454	618	764	887	984
24	1	13	61	169	333	532	733	919	1076	1204
25	1	13	65	185	377	612	860	1090	1291	1455
26	1	14	70	206	427	709	1009	1297	1549	1761
27	1	14	75	225	480	811	1175	1527	1845	2112
28	1	15	80	249	540	931	1367	1801	2194	2534
29	1	15	85	270	603	1057	1579	2104	2592	3015
30	1	16	91	297	674	1206	1824	2462	3060	3590

This may be illustrated by the partition  $(6^2 4 3 2)$  which has the Young diagram



and hence the partition is of rank 3 and in the Frobenius notation is designated as

$$\begin{pmatrix} 5 & 4 & 1 \\ 4 & 3 & 1 \end{pmatrix}$$

Note that the numbers in each row strictly decrease, and that zeros, even two zeros one above the other, are significant.

The two examples, of the Frobenius notation given below

$$\begin{pmatrix} 5 & 4 & 0 \\ 5 & 3 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 4 \\ 5 & 3 \end{pmatrix}$$

represent different partitions.

Another concept called the hook plays an important part in the theory of irreps of  $S_n$ . In Young diagram  $(\lambda)$  the square located in  $i$ -row and  $j$ -column is called  $(i,j)$ -node of  $(\lambda)$  and the  $(i,j)$ -hook of  $(\lambda)$  is a  $\Gamma$ -shaped arrangement of nodes out of a Young diagram  $(\lambda)$  which consists of the

$(i,j)$ -nodes along with the  $\lambda_i - j$  nodes to the right of it (called the arm of the hook) and the  $\lambda_j' - i$  nodes below it (called the leg of the hook). The length of the  $(i,j)$ -hook is  $h_{ij}^{(\lambda)} = \lambda_i + \lambda_j' + 1 - i - j$ . If we replace the  $(i,j)$ -node of  $(\lambda)$  by the number  $h_{ij}^{(\lambda)}$  for each node, we obtain the hook graph.

For example, the  $(2,2)$ -hook of the Young diagram  $(43^21)$  is

	x	x		
	x			

and the hook graph is

7	5	4	1
5	3	2	
4	2	1	
1			

The formula, due to Frame, Robinson and Thrall (Frame et al. 1954), for the dimension of irrep  $[\lambda]$  of  $S_n$  in terms of the hook graph is of fascinating simplicity. The dimension of irrep  $[\lambda]$  is

$$f^{[\lambda]} = n!/H(\lambda) \quad (1.20)$$

where  $H(\lambda) = \prod_{ij} h_{ij}^{(\lambda)}$  called the product of the hook lengths.

Thus the dimension of irrep  $[43^21]$  of  $S_{11}$  is

$$f^{[43^21]} = 11!/7 \times 5^2 \times 4^4 \times 3 \times 2^2 = 1188$$



From the above discussion we know that  $S_n$  has two nonequivalent one dimensional irreps  $[n]$  and  $[1^n]$ .  $[n]$  corresponds to the identity irrep and  $[1^n]$  to the alternating irrep. We also have the fact that  $[1^n] \cdot [\lambda]$  is an irrep and it will be said to be the associated irrep of  $[\lambda]$ . It is easy to prove it's Young diagram is  $(\tilde{\lambda})$  so we use  $[\tilde{\lambda}]$  to label  $[1^n] \cdot [\lambda]$ .

If  $[\tilde{\lambda}] \cong [\lambda]$  the  $[\lambda]$  will be called self-associated and denoted by  $[\lambda]^\dagger$ .

If  $[\tilde{\lambda}] \neq [\lambda]$ , we note that  $[\lambda] + [\tilde{\lambda}]$  corresponds to a reducible self-associated rep of  $S_n$ . As a consequence we shall often use  $[\lambda]^\dagger$  with the understanding that if  $[\tilde{\lambda}] \neq [\lambda]$  then

$$[\lambda]^\dagger = [\lambda] + [\tilde{\lambda}] \quad (1.21)$$

#### result

Here we would like to quote a/due to Littlewood (1950) (cf. Rutherford, 1948) which will be used in the following chapter.

#### Theorem.

The character  $\chi_{(\rho)}^{[\pi]}$  is given by

$$\chi_{(\rho)}^{[\pi]} = \sum d_i \quad (1.22)$$

where there is one term  $d_i$  for each way in which the shape  $(\pi)$  can be built up by making firstly a regular application of  $\rho_1$  spaces, secondly a regular application of  $\rho_2$  spaces,

..., and lastly a regular application of  $\rho_h$  spaces, and where  $d_i = (-1)^{t_i}$ ,  $t_i$  being the sum of the numbers of vertical steps in the  $h$  applications.

#### 1.4 THE SPIN REPRESENTATIONS OF $S_n$

In Sec. 1.2 we pointed out that the  $H^2(S_n, C^X) = C_2 = \{r\}$ . Correspondingly  $r = -1$ , the i.r.p. of  $S_n$  also called the spin irreps. The spin irreps of  $S_n$  may be uniquely labelled by the ordered partitions of  $n$  into unequal parts (Schur 1911) i.e.

$$\lambda_1 > \lambda_2 > \dots > \lambda_k > 0 \quad (1.23a)$$

with

$$\lambda_1 + \lambda_2 + \dots + \lambda_k = n \quad (1.23b)$$

To distinguish spin irreps from ordinary irreps of  $S_n$  we shall use a prime. Thus  $[531]'$  is a spin irrep while  $[531]$  is an ordinary irrep of  $S_9$ .

The Kronecker product of ordinary irreps with spin irreps are also spin reps (Altmann et al., 1979), so  $[1^n][\lambda]'$  is a spin irrep of  $S_n$ , whose dimension is the same as the  $[\lambda]'$ , denoted by  $[\tilde{\lambda}]'$ . If  $[\tilde{\lambda}]' \cong [\lambda]'$  it be also called self-associated denoted by  $[\lambda]'^{\dagger}$  and if  $[\tilde{\lambda}]' \neq [\lambda]'$  then  $[\tilde{\lambda}]'$  will be called the associated irrep of  $[\lambda]'$ .

In fact if  $(n-k)$  is even the irrep is self-associated while if  $(n-k)$  is odd we obtain an associated pair of spin irrep. We note that  $[\lambda]' + [\tilde{\lambda}]'$  corresponds to a reducible

self-associated rep for  $(n-k)$  odd. As a consequence we shall often use  $[\lambda]'^{\dagger}$  without regard to the parity of  $(n-k)$  with the understanding that if  $(n-k)$  is odd then

$$[\lambda]'^{\dagger} = [\lambda]' + [\tilde{\lambda}]' \quad (n-k) \text{ odd} \quad (1.24)$$

The dimension formula for spin irreps of  $S_n$  was given long ago (Schur 1911) for a  $k$  parts partition  $(\lambda_1, \lambda_2, \dots, \lambda_k)$  as

$$f^{[\lambda]'} = 2^{[(n-k)/2]} n! \prod_{i=1}^k (\lambda_i!)^{-1} \prod_{1 \leq \ell < s \leq k} \left( \frac{\lambda_{\ell} - \lambda_s}{\lambda_{\ell} + \lambda_s} \right) \quad (1.25)$$

where  $[(n-k)/2]$  denotes the greatest integer  $\leq (n-k)/2$ .

For each value of  $n$  there is a basic spin irrep  $[n]'^{\dagger}$  of dimension  $2^{[n/2]}$  where  $[x]$  denotes the greatest integer less than equal to  $x$ .

For the spin irreps of  $S_n$ , (1.12) is satisfied, so the characters are the class functions and vanish on  $\alpha$ -irregular class. We only consider the value of characters on  $\alpha$ -regular classes. First, considering odd  $\alpha$ -regular classes  $(\lambda_1 \lambda_2 \dots \lambda_k) (\lambda_1 > \lambda_2 > \dots \lambda_k)$ . For  $(n-k)$  the value of character of irrep  $[\lambda_1 \lambda_2 \dots \lambda_k]'$  on class  $(\lambda_1 \lambda_2 \dots \lambda_k)$  is non-zero given by

$$\chi_{(\lambda)}^{[\lambda]'} = i^{(n-k-1)/2} (\lambda_1 \lambda_2 \dots \lambda_k / 2)^{\frac{1}{2}} \quad (1.26a)$$

and

$$\chi_{(\lambda)}^{[\tilde{\lambda}]'} = - \chi_{(\lambda)}^{[\lambda]'} \quad (1.26b)$$

while its value on every other odd  $\alpha$ -regular classes are zero.

In dealing with the associated spin irrep when  $(n-k)$  is odd it is useful to exploit the properties of difference characters

$$\chi^{[\lambda]'''} = \chi^{[\lambda]'} - \chi^{[\tilde{\lambda}]'} \quad (n-k) \text{ odd} \quad (1.27)$$

Noting (1.24) we have

$$\chi^{[\lambda]'} = (\chi^{[\lambda]'}{}^\dagger + \chi^{[\lambda]''})/2 \quad (1.28a)$$

and

$$\chi^{[\tilde{\lambda}]'} = (\chi^{[\lambda]'}{}^\dagger - \chi^{[\lambda]''})/2 \quad (1.28b)$$

For the odd  $\alpha$ -regular class  $(\lambda_1 \lambda_2 \cdots \lambda_k)$  we have the difference character

$$\chi_{(\rho)}^{[\lambda]'''} = i^{(n-k+1)/2} (2^{\lambda_1 \lambda_2 \cdots \lambda_k})^{\frac{1}{2}} \quad (1.29)$$

The difference character for all other classes is zero.

For even  $\alpha$ -regular classes, Morris has established the results for  $n \leq 13$  as follows (Morris, 1962a).

Let

$$[\lambda]' = [\lambda_1 \lambda_2 \cdots \lambda_k]'$$

$$(\pi) = (\cdots \overset{\alpha_5}{5} \quad \overset{\alpha_3}{3} \quad \overset{\alpha_1}{1} \quad )$$

and

$$p = \alpha_1 + \alpha_3 + \alpha_5 + \cdots$$

then we have

$$(1) \quad k = 2$$

$$\chi_{(\pi)}^{[n-1,1]'} = 2^{(p-2-t)/2} (\alpha_1 - 2),$$

$$\chi_{(\pi)}^{[n-2,2]'} = 2^{(p-2-t)/2} \frac{(\alpha_1 - 1)(\alpha_1 - 4)}{2!}$$

$$\chi_{(\pi)}^{[n-3,3]'} = 2^{(p-2-t)/2} \left\{ \frac{(\alpha_1 - 1)(\alpha_1 - 2)(\alpha_1 - 6)}{3!} + \alpha_3 \right\}$$

$$\chi_{(\pi)}^{[n-4,4]'} = 2^{(p-2-t)/2} \left\{ \frac{(\alpha_1 - 1)(\alpha_1 - 2)(\alpha_1 - 3)(\alpha_1 - 8)}{4!} + (\alpha_1 - 2)\alpha_3 \right\}$$

$$\begin{aligned} \chi_{(\pi)}^{[n-5,5]'} &= 2^{(p-2-t)/2} \left\{ \frac{(\alpha_1 - 1)(\alpha_1 - 2)(\alpha_1 - 3)(\alpha_1 - 4)(\alpha_1 - 10)}{5!} \right. \\ &\quad \left. + \frac{\alpha_1 \alpha_3 (\alpha_1 - 5)}{2} + 2\alpha_3 + \alpha_5 \right\}, \end{aligned}$$

$$(2) \quad k = 3$$

$$\chi_{(\pi)}^{[n-3,21]'} = 2^{(p-3-t)/2} \left\{ \frac{\alpha_1 (\alpha_1 - 4)(\alpha_1 - 5)}{3!} - 2\alpha_3 \right\},$$

$$\chi_{(\pi)}^{[n-4,31]'} = 2^{(p-3-t)/2} \left\{ \frac{2\alpha_1 (\alpha_1 - 2)(\alpha_1 - 5)(\alpha_1 - 7)}{4!} - \alpha_3 (\alpha_1 - 2) \right\},$$

$$\chi_{(\pi)}^{[n-5,41]'} = 2^{(p-3-t)/2} \left\{ \frac{3\alpha_1 (\alpha_1 - 2)(\alpha_1 - 3)(\alpha_1 - 6)(\alpha_1 - 9)}{5!} - 2\alpha_5 \right\},$$

$$\begin{aligned} \chi_{(\pi)}^{[n-6,51]'} &= 2^{(p-3-t)/2} \left\{ \frac{4\alpha_1 (\alpha_1 - 2)(\alpha_1 - 3)(\alpha_1 - 4)(\alpha_1 - 7)(\alpha_1 - 11)}{6!} \right. \\ &\quad \left. + \frac{\alpha_1 \alpha_3 (\alpha_1 - 4)(\alpha_1 - 5)}{3!} - \alpha_5 (\alpha_1 - 2) - \alpha_3 (\alpha_3 - 1) \right\}, \end{aligned}$$

$$\begin{aligned} \chi_{(\pi)}^{[n-5,32]'} &= 2^{(p-3-t)/2} \left\{ \frac{2\alpha_1 (\alpha_1 - 1)(\alpha_1 - 4)(\alpha_1 - 7)(\alpha_1 - 8)}{5!} \right. \\ &\quad \left. - \frac{\alpha_1 \alpha_3 (\alpha_1 - 5)}{2!} - 2\alpha_3 + 2\alpha_5 \right\}, \end{aligned}$$

$$\begin{aligned}
\chi_{(\pi)}^{[n-6,421]'} &= 2^{(p-3-t)/2} \left\{ \frac{5\alpha_1(\alpha_1-1)(\alpha_1-3)(\alpha_1-5)(\alpha_1-8)(\alpha_1-10)}{6!} \right. \\
&\quad \left. + \frac{\alpha_1\alpha_3(\alpha_1-4)(\alpha_1-5)}{3!} + \alpha_3(\alpha_3-1) \right\}, \\
\chi_{(\pi)}^{[n-7,431]'} &= 2^{(p-3-t)/2} \left\{ \frac{5\alpha_1(\alpha_1-1)(\alpha_1-2)(\alpha_1-5)(\alpha_1-6)(\alpha_1-10)(\alpha_1-11)}{7!} \right. \\
&\quad \left. - \frac{\alpha_1\alpha_3(\alpha_1-2)(\alpha_1-5)(\alpha_1-7)}{4!} + \alpha_3(\alpha_3-2)(\alpha_1-2) - 2\alpha_7 \right\}
\end{aligned}$$

(3)  $k = 4$

$$\begin{aligned}
\chi_{(\pi)}^{[n-6,321]'} &= 2^{(p-4-t)/2} \left\{ \frac{2\alpha_1(\alpha_1-1)(\alpha_1-2)(\alpha_1-7)(\alpha_1-8)(\alpha_1-9)}{6!} \right. \\
&\quad \left. + \frac{\alpha_1\alpha_3(\alpha_1-4)(\alpha_1-5)}{3!} - 2\alpha_3(\alpha_3-3) - 4\alpha_5 + 2\alpha_1\alpha_5 \right\}, \\
\chi_{(\pi)}^{[n-7,421]'} &= 2^{(p-4-t)/2} \left\{ \frac{\alpha_1(\alpha_1-1)(\alpha_1-2)(\alpha_1-4)(\alpha_1-8)(\alpha_1-9)(\alpha_1-11)}{6!} \right. \\
&\quad \left. - \frac{2\alpha_1\alpha_3(\alpha_1-2)(\alpha_1-5)(\alpha_1-7)}{4!} \right. \\
&\quad \left. - \alpha_3(\alpha_3-2)(\alpha_1-2) + \alpha_1\alpha_5(\alpha_1-5) + 4\alpha_5 \right\},
\end{aligned}$$

where in each case  $t = 0$  if  $n$  is even,  $t = 1$  if  $n$  is odd.

As an example, we give the table of spin character for the  $\alpha$ -regular classes of  $S_8$  in table II.

The Kronecker product of a spin irrep with a spin irrep is an ordinary rep as well the Kronecker product of a spin irrep with an ordinary irrep is a spin rep (Altmann et al., 1979), so the basic spin irrep is such an irrep from which every irrep of  $S_n$  arise in a Kronecker power of the basic spin irrep.

TABLE II: Table of spin character for  $\alpha$ -regular classes of  $S_8$

Class	$(1^8)$	$(31^5)$	$(51^3)$	$(3^21^2)$	$(71)$	$(53)$	$(8)$	$(521)$	$(431)$
Order	1	112	1344	1120	5760	2688	5040	4032	3360
$[8]'$	8	4	2	2	1	1	2	0	0
$[8]''$	8	4	2	2	1	1	-2	0	0
$[71]'^{\dagger}$	48	12	2	0	-1	-2	0	0	0
$[62]'^{\dagger}$	112	8	-2	-2	0	2	0	0	0
$[53]'^{\dagger}$	112	-4	-2	4	0	-1	0	0	0
$[521]'$	64	-4	1	-2	1	-1	0	$\sqrt{5}i$	0
$[521]''$	64	-4	1	-2	1	-1	0	$-\sqrt{5}i$	0
$[431]'$	48	-6	2	0	-1	1	0	0	$-\sqrt{6}i$
$[431]''$	48	-6	2	0	-1	1	0	0	$-\sqrt{6}i$

### 1.5 PROPERTIES OF ASSOCIATED AND SELF-ASSOCIATED REPRESENTATIONS

In the previous sections we have noted that both the ordinary and spin irreps of  $S_n$  may be divided into two classes: associated and self-associated irreps. This is a general property of groups that contain the one-dimensional alternating irrep.

Two theorems concerning associated and self-associated irreps play an important role in our subsequent analysis of the properties of the irreps of  $S_n$  both in ordinary and spin cases. These theorems reinforce the usefulness of the notation developed in the previous sections.

#### Theorem 1

If a group  $G$  contains a subgroup  $H$  with  $\lambda_i^+$ ,  $\lambda_i$ ,  $\tilde{\lambda}_i$  and  $\rho_i^+$ ,  $\rho_i$ ,  $\tilde{\rho}_i$  being their respective self-associated and associated pair irreps then under  $G \downarrow H$

$$(i) \quad \lambda_i^+ \downarrow a_i^j \rho_j^+ + b_i^j (\rho_j + \tilde{\rho}_j) \quad (1.30)$$

$$(ii) \quad \text{if } \lambda_i \downarrow a_i^j \rho_j^+ + b_i^j \rho_j$$

then

$$\tilde{\lambda}_i \downarrow a_i^j \rho_j^+ + b_i^j \tilde{\rho}_j \quad (1.31)$$

where the coefficients  $a_i^j$  and  $b_i^j$  are non-negative integer multiplicity numbers.



Proof

(i) Suppose

$$\lambda_i^+ + a_{i\rho_j}^j + b_{i\rho_j}^j + c_{i\tilde{\rho}_j}^j$$

For  $\lambda_i^+$  is self-associated,  $\lambda_i^+|_H$  as the subset of  $\lambda_i^+$  is also self-associated hence

$$\lambda_i^+|_H = \tilde{\lambda}_i^+|_H$$

i.e.

$$a_{i\rho_j}^j + b_{i\tilde{\rho}_j}^j + c_{i\rho_j}^j = a_{i\rho_j}^j + b_{i\tilde{\rho}_j}^j + c_{i\rho_j}^j$$

Compare the coefficients of the two sides of the equation we have

$$b_i^j = c_i^j$$

so

$$\lambda_i^+ + a_{i\rho_j}^j + b_i^j(\rho_j + \tilde{\rho}_j)$$

(ii) the proof is trivial.

Theorem 2

Let  $\lambda_i^+$ ,  $\lambda_i$ ,  $\tilde{\lambda}_i$  be self-associated and associated pair irreps of G. The Kronecker products of the irreps of G necessarily satisfy the identities

$$(i) \quad \lambda_i^+ \times \lambda_j = \lambda_i^+ \times \tilde{\lambda}_j = a_{ij}^k \lambda_k^+ + b_{ij}^k (\lambda_k + \tilde{\lambda}_k) \quad (1.32)$$

$$(ii) \quad \lambda_i^\dagger \times \lambda_j^\dagger = a_{ij}^k \lambda_k^\dagger + b_{ij}^k (\lambda_k + \tilde{\lambda}_k) \quad (1.33)$$

$$(iii) \quad \tilde{\lambda}_i \times \tilde{\lambda}_j = \lambda_i \times \lambda_j \quad (1.34)$$

$$(iv) \quad \text{if } \lambda_i \times \lambda_j = a_{ij}^k \lambda_k^\dagger + b_{ij}^k \lambda_k$$

then

$$\lambda_i \times \tilde{\lambda}_j = \tilde{\lambda}_i \times \lambda_j = a_{ij}^k \lambda_k^\dagger + b_{ij}^k \tilde{\lambda}_k \quad (1.35)$$

The proof of this theorem is as similar as the  
theorem 1.

to that of

## CHAPTER 2

### THE SYMMETRIC FUNCTIONS

#### 2.1 THE BASIC SYMMETRIC FUNCTIONS

The theory of symmetric functions play an important role in our later discussions. Here we give some facts about the theory of symmetric functions (Littlewood, 1958, Wybourne 1970).

Consider a set of countably many variables (finite or infinite)

$$(x) \equiv \{x_1, x_2, \dots\} \quad (2.1)$$

Any product of powers of  $(x)$  such that

$$x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \dots \quad \lambda_1, \lambda_2, \lambda_3, \dots \in \mathbb{Z}^+ \quad (2.2)$$

will be termed a monomial function.

From monomials of (2.2) we can define some basic symmetric functions.

Monomial symmetric functions are those functions which are formed from the monomials of (2.2) and which remain unchanged under any permutation of variables  $(x)$ . Thus in three variables examples of monomial symmetric functions would be

$$\sum x_i = x_1 + x_2 + x_3$$

$$\sum x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3$$

$$\sum x_i^2 = x_1^2 + x_2^2 + x_3^2$$

$$\sum x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2$$

etc.

where the summation sign indicates that all distinct monomials in the  $x_i$ 's, with the exponents in a prescribed order, are to be included.

It is apparent from the above simple examples that we can define a monomial symmetric function  $k_{(\lambda)}$  for each partition  $(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_p)$  of  $n$  such that

$$k_{(\lambda)} = k_{(\lambda)}(x) = \sum x_1^{\lambda_1} x_2^{\lambda_2} \dots x_p^{\lambda_p} \quad (2.3)$$

where the summation is over all monomials obtained from  $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_p^{\lambda_p}$  by a permutation of the variables  $(x)$  e.g.

$$k_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2$$

particular choices of the partition  $(\lambda)$  lead to special cases of symmetric functions.

Now consider some cases

$$(i) \quad \text{if } (\lambda) = (1^r)$$

then

$$a_r = k_{(1^r)}(x) \quad (2.4)$$

is called an elementary symmetric function. Clearly

$a_r (r \geq 0)$  is the sum of all products of  $n$  distinct variables  $x_i$ . For  $r = 0$ ,  $a_0 = 1$  while if  $r > 0$

$$a_r(x) = \sum x_1 x_2 \dots x_r \quad (2.5)$$

and we define

$$a_{(\lambda)} = a_{\lambda_1}(x) a_{\lambda_2}(x) \dots a_{\lambda_p}(x) \quad (2.6)$$

for  $(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_p)$ . In three variables we have

$$a_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$$

$$a_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

while

$$\begin{aligned} a_{(21)}(x_1, x_2, x_3) &= (x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3) \\ &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2 + 3x_1 x_2 x_3 \end{aligned}$$

The elementary symmetric functions  $a_r(x)$  are associated with the generating function

$$A(t) = \sum_{r=0}^{\infty} a_r t^r = \prod_{i=1}^{\infty} (1 + x_i t) \quad (2.7)$$

(ii) if  $(\lambda) = (r)$

then

$$s_r = k_{(r)}(x) \quad (2.8)$$

is called the power sum symmetric function which clearly is sum of the  $n$ -th powers of  $x_i$ 's i.e.

$$s_r(x) = \sum x_i^r \quad (2.9)$$

and we define

$$s_{(\lambda)} = s_{\lambda_1} s_{\lambda_2} \dots s_{\lambda_p} \quad (2.10)$$

for  $(\lambda) = (\lambda_1 \lambda_2 \dots \lambda_p)$ .

For example, we have

$$s_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

$$s_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

while

$$\begin{aligned} s_{(21)}(x_1, x_2, x_3) &= (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3) \\ &= x_1^3 + x_1x_2^2 + x_1x_3^2 + x_1^2x_2 + x_2^3 + x_2x_3^2 + x_1^2x_3 \\ &\quad + x_2^2x_3 + x_3^3 \end{aligned}$$

The power sum symmetric functions are associated with the generating function

$$\begin{aligned} s(t) &= \sum_{r=1}^{\infty} s_r t^{r-1} = \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} x_i^r t^{r-1} \\ &= \sum_{i=1}^{\infty} \frac{x_i}{1-x_i t} \\ &= \sum_{i=1}^{\infty} \frac{d}{dt} \lg \frac{1}{1-x_i t} \end{aligned} \quad (2.11)$$

(iii) The sum of all monomial symmetric functions of degree  $r$  in the  $x_i$ 's is called the complete symmetric function or homogeneous product sum denoted by  $h_r$  i.e.

$$h_r = \sum_{\lambda \vdash r} k_{(\lambda)} \quad (2.12)$$

where  $\lambda \vdash r$  denotes all the partitions of  $(\lambda)$  of  $r$  and define

$$h_{(\lambda)} = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_p} \quad (2.13)$$

for  $(\lambda) = (\lambda_1 \lambda_2 \dots \lambda_p)$ . Thus

$$h_0 = 1$$

$$h_1 = \sum x_1$$

$$h_2 = \sum x_1^2 + \sum x_1 x_2$$

$$h_3 = \sum x_1^3 + \sum x_1^2 x_2 + \sum x_1 x_2 x_3$$

etc.

and

$$\begin{aligned} h_{(21)} &= (\sum x_1) (\sum x_1^2 + \sum x_1 x_2) \\ &= \sum x_1^3 + 2 \sum x_1^2 x_2 + 3 \sum x_1 x_2 x_3 \end{aligned}$$

The complete symmetric functions  $h_r(x)$  are associated with the generating function

$$H(t) = \sum_{r=0}^{\infty} h_r t^r = \prod_{i=1}^{\infty} (1 - x_i t)^{-1} \quad (2.14)$$

## 2.2 THE RELATIONS BETWEEN SYMMETRIC FUNCTIONS

The  $a_r$  and  $h_r$  may be simply related by noting that their generating functions lead to the identity

$$H(t)A(t) = 1 \quad (2.15)$$

and hence

$$\sum_{r=0}^n (-1)^r a_r h_{n-r} = 0 \quad (n=1,2,3, \dots) \quad (2.16)$$

leading to the determinantal forms

$$a_r = \begin{vmatrix} h_1 & 1 \\ h_2 & h_1 & 1 \\ h_3 & h_2 & h_1 & 1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 \\ h_r & h_{r-1} & \dots & \dots & h_1 \end{vmatrix} \quad (2.17)$$

and

$$h_r = \begin{vmatrix} a_1 & 1 \\ a_2 & a_1 & 1 \\ a_3 & a_2 & a_1 & 1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 \\ a_r & a_{r-1} & \dots & \dots & a_1 \end{vmatrix} \quad (2.18)$$

In a similar way we find

$$nh_n = \sum_{r=1}^n s_r h_{n-r} \quad (n=1,2,3, \dots) \quad (2.19)$$

leading to the relationships



$$s_r = (-1)^{r-1} \begin{vmatrix} h_1 & 1 & & & \\ 2h_2 & h_1 & & & \\ 3h_3 & h_2 & h_1 & 1 & \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & 1 \\ rh_r & h_{r-1} & \dots & \dots & h_1 \end{vmatrix}$$

and

$$r!h_r = \begin{vmatrix} s_1 & -1 & & & \\ s_2 & s_1 & -2 & & \\ s_3 & s_2 & s_1 & -3 & \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & -r+1 \\ s_r & s_{r-1} & \dots & \dots & s_1 \end{vmatrix} \quad (2.21)$$

Likewise we also have

$$s_r = \begin{vmatrix} a_1 & 1 & & & \\ 2a_2 & a_1 & 1 & & \\ 3a_3 & a_2 & a_1 & 1 & \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & 1 \\ ra_r & a_{r-1} & \dots & \dots & a_1 \end{vmatrix} \quad (2.22)$$

and

$$r!a_r = \begin{vmatrix} s_1 & 1 & & & \\ s_2 & s_1 & 2 & & \\ s_3 & s_2 & s_1 & 3 & \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & 1 \\ s_r & s_{r-1} & \dots & \dots & a_1 \end{vmatrix} \quad (2.23)$$

### 2.3 SCHUR FUNCTIONS

There are several equivalent definitions about Schur functions or S-functions. The traditional definition is in terms of determinantal expansions involving the variables  $(x)$ . Let  $(\lambda) = (\lambda_1 \lambda_2 \dots \lambda_p)$  is a partition of  $r$  then in  $n$  variable  $(x)$  of the S-function  $\{\lambda\}$  is defined by (Littlewood, 1950)

$$\{\lambda\} = \frac{|x_j^{\lambda_i + n - i}|}{|x_j^{n - i}|} \quad (2.24)$$

For  $\{21\}(x_1, x_2, x_3)$  we have

$$\begin{aligned} \{21\} &= \frac{\begin{vmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1^2 & x_2^2 & x_3^2 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}} \\ &= \frac{x_1^4(x_2^2 - x_3^2) + x_2^4(x_3^2 - x_1^2) + x_3^4(x_1^2 - x_2^2)}{x_1^2(x_2 - x_3) + x_2^2(x_3 - x_1) + x_3^2(x_1 - x_2)} \\ &= \Sigma x_1^2 x_2 + 2 \Sigma x_1 x_2 x_3 \end{aligned}$$

Equation (2.24) is cumbersome in practice. However, the S-function  $\{\lambda\}$  are symmetric functions and hence may be related to the symmetric functions  $a_{(\lambda)}$ ,  $h_{(\lambda)}$  and  $S_{(\lambda)}$ . In fact, there exists the remarkable Jacobi-Trudi identity (Littlewood 1958)

$$\frac{|x_j^{\lambda_i + n - i}|}{|x_j^{n - i}|} = |h_{\lambda_i - i + j}| \quad (2.25)$$

which in (2.24) gives

$$\{\lambda\} = |h_{\lambda_i - i + j}| \quad (2.26)$$

with  $h_r = 0$  if  $r < 0$ .

From (2.26) we can define the dimension of  $\{\lambda\}, f^\lambda$

$$f^{(\lambda)} = w_\lambda! \left| \frac{1}{(\lambda_i - i + j)!} \right| \quad (2.27)$$

This is the dimension of irrep  $[\lambda]$  of  $S_N$  (James, 1978).

Thus

$$\{r\} = h_r \quad (2.28)$$

and

$$\begin{aligned} \{21\} &= \begin{vmatrix} h_2 & h_3 \\ 1 & h_1 \end{vmatrix} = h_1 h_2 - h_3 \\ &= \sum x_1 (\sum x_1 x_2 + \sum x_1^2) - (\sum x_1^3 + \sum x_1^2 x_2 + \sum x_1 x_2 x_3) \end{aligned}$$

But

$$\sum x_1 (\sum x_1 x_2) = \sum x_1^2 x_2 + 3 \sum x_1 x_2 x_3$$

and

$$\sum x_1 (\sum x_1^2) = \sum x_1^3 + \sum x_1^2 x_2$$

hence we have again

$$\{21\} = \sum x_1^2 x_2 + 2 \sum x_1 x_2 x_3$$

If  $(\tilde{\lambda}) = (\mu)$  then (Littlewood, 1950)

$$\{\lambda\} = |a_{\mu_i - i + j}| \quad (2.29)$$

with  $a_r = 0$  if  $r < 0$ .

Thus

$$\{1^r\} = a_r \quad (2.30)$$

and

$$\begin{aligned} \{21\} &= \begin{vmatrix} a_2 & a_3 \\ 1 & a_1 \end{vmatrix} = a_1 a_2 - a_3 \\ &= \sum x_1 (\sum x_1 x_2) - \sum x_1 x_2 x_3 \\ &= \sum x_1^2 x_2 + 2 \sum x_1 x_2 x_3 \end{aligned}$$

It has been assumed in the definition of the S-function  $\{\lambda\}$  that the parts  $\lambda_i$  of  $(\lambda)$  are in standard descending order and non-negative. It is convenient to be able to define the S-function  $\{\lambda\}$  even when the parts are not in standard order or may be negative. In these cases consideration of the determinantal expression (2.26) leads to a set of three rules for transforming a non-standard S-function into standard form (Littlewood, 1950). The rules are

$$\begin{aligned} 1. \quad & \{\lambda_1, \lambda_2, \dots, \lambda_i, \lambda_{i+1}, \dots, \lambda_p\} \\ &= - \{\lambda_1, \lambda_2, \dots, \lambda_{i+1}-1, \lambda_i+1, \dots, \lambda_p\} \end{aligned} \quad (2.31a)$$

$$2. \quad \text{If } \lambda_{i+1} = \lambda_i + 1 \text{ then } \{\lambda\} = 0 \quad (2.31b)$$

$$3. \quad \text{If } \lambda_p < 0 \text{ then } \{\lambda\} = 0 \quad (2.31c)$$

which are called modification rules of S-function.

For example

$$\{2332\} = - \{2242\} = \{1342\} = 0$$

(Using (2.31a) and (2.31b) and

$$\{2 \ 5 \ -2 \ 3\} = - \{2 \ 5 \ 4 \ -3\} = 0$$

(using (2.31a) and (2.31c)).

## 2.4 RAISING OPERATORS

In the previous section we gave the relationships between  $\{\lambda\}$  and other symmetric functions such as in (2.26) and (2.29). Here we give a raising operator method to express  $\{\lambda\}$  by  $h_{(\lambda)}$  or vice versa. (Thomas 1976).

The Raising Operator  $R_{ij}$  ( $i < j$ ) is an operator which operates on a partition  $(\lambda)$  by increasing  $\lambda_i$  by one and decreasing  $\lambda_j$  by one so that

$$\begin{aligned} R_{ij}(\lambda_1, \lambda_2, \dots, \lambda_i, \lambda_j, \dots) \\ = (\lambda_1, \lambda_2, \dots, \lambda_i+1, \lambda_j-1, \dots) \end{aligned} \quad (2.32)$$

Consideration of (2.32) and (2.26) leads to the remarkable results

$$\{\lambda\} = \prod_{i < j} (1 - R_{ij}) h_{\lambda} \quad (2.33)$$

and

$$\begin{aligned} h_{\lambda} &= \prod_{i < j} \frac{1}{1 - R_{ij}} \{\lambda\} \\ &= \prod_{i < j} (1 + R_{ij} + R_{ij}^2 + \dots) \{\lambda\} \end{aligned} \quad (2.34)$$

$$(1+R_{13}+R_{13}^2+\dots)(\{321\} + \{330\})$$

$$= \{321\} + \{330\} + \{420\}$$

and then

$$(1+R_{12}+R_{12}^2+\dots)(\{321\} + \{330\} + \{42\})$$

$$= \{321\} + \{411\} + \{501\} + \{6-11\} + \{33\} + \{42\} + \{51\}$$

$$+ \{6\} + \{42\} + \{51\} + \{6\}$$

But  $\{501\} = 0$  and  $\{6-11\} = -\{6\}$  and hence

$$h_{321} = \{6\} + 2\{51\} + 2\{42\} + \{41^2\} + \{3^2\} + \{321\}.$$

## 2.5 THE OUTER PRODUCT OF S-FUNCTIONS

The outer product of two S-functions  $\{\mu\}$  and  $\{\nu\}$  may be defined by writing

$$\{\mu\nu\}(x) = \{\mu\}(x) \cdot \{\nu\}(x) \quad (2.35)$$

which may be represented as a determinant in the symmetric functions  $h_r$ . Putting

$$r = p_\mu + p_\nu \quad (2.36)$$

we have

$$\{\mu \cdot \nu\} = |h_{\mu_i + \nu_{r+1-j} - i + j}| \quad (2.37)$$

where the determinant is  $r \times r$  and its dimension is defined by

$$f^{(\mu\nu)} = (w_\mu + w_\nu)! \left| \frac{1}{(\mu_i + \nu_{r+1-j} - i + j)!} \right| \quad (2.38)$$

It is equal to

$$f^{(\mu\nu)} = \frac{(w_\mu + w_\nu)!}{w_\mu! w_\nu!} f^{(\mu)} f^{(\nu)} \quad (2.39)$$

For example

$$\{21 \cdot 21^2\} = \begin{vmatrix} h_2 & h_3 & h_5 & h_6 & h_8 \\ 1 & h_1 & h_3 & h_4 & h_6 \\ 0 & 0 & h_1 & h_2 & h_4 \\ 0 & 0 & 1 & h_1 & h_3 \\ 0 & 0 & 0 & 1 & h_2 \end{vmatrix}$$

$$= h_{421} + h_{3^2 1} + h_{32^2} + h_{2^2 1^3} - h_{43} - 2h_{321^2} - h_{2^3 1}$$

a result that could also be obtained by use of (2.26) by multiplication of the determinantal expansions of  $\{21\}$  and  $\{21^2\}$  and

$$f^{(21 \cdot 21^2)} = 210$$

The product so defined is a symmetric function and may be expressed as the sum of S-functions using (2.34) to give

$$\{\mu \cdot \nu\} = g_{\mu\nu}^\lambda \{\lambda\} \quad (2.40)$$

The coefficients  $g_{\mu\nu}^\lambda$  are positive integers and may be evaluated by the well-known Littlewood-Richardson rule (Littlewood (1958), but we will give another procedure due to Wybourne in section 2.7.

Hence if  $(\lambda) = (321)$  then (2.32) leads to

$$\begin{aligned}
 \{321\} &= (1-R_{12})(1-R_{13})(1-R_{23})h_{321} \\
 &= (1-R_{12})(1-R_{13})(h_{321}-h_{330}) \\
 &= (1-R_{12})(h_{321}-h_{33}-h_{42}) \\
 &= h_{321}-h_{33}-h_{42}-h_{411}+h_{42}+h_{51} \\
 &= h_{321}-h_{33}-h_{411}+h_{51} \\
 &= h_5h_1 + h_3h_2h_1 - h_3^2 - h_4h_1^2
 \end{aligned}$$

which is readily seen to be equivalent to the expansion of the determinant

$$\begin{vmatrix} h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix}$$

obtained from (2.26).

Likewise use of (2.33) leads to

$$\begin{aligned}
 h_{321} &= (1+R_{12}+R_{12}^2+\dots)(1+R_{13}+R_{13}^2+\dots) \\
 &\quad (1+R_{23}+R_{23}^2+\dots)\{321\}
 \end{aligned}$$

But

$$(1+R_{23}+R_{23}^2+\dots)\{321\} = \{321\} + \{330\}$$

with higher terms such as  $\{34-1\}$  all being null.



Equation (2.40) can be checked by the dimensional relations

$$f^{(\mu\nu)} = g_{\mu\nu}^{\lambda} f^{(\lambda)} \quad (2.41)$$

## 2.6 THE SKEW S-FUNCTIONS

Let  $(\lambda)$  is the partition of  $w_{\lambda}$  and  $(\mu)$  the partition of  $w_{\mu}$  ( $w_{\lambda} > w_{\mu}$ ).  $w_{\lambda} = w_{\mu}$  possible if  $(\lambda) \equiv (\mu)$  to give  $(\lambda/\mu) = (0)$ . A skew S-function,  $\{\lambda/\mu\}$ , may be defined analogously to that of  $\{\lambda\}$  by the generalization of Jacobi-Trudy identity

$$\{\lambda/\mu\} = |h_{\lambda_i - \mu_j - i + j}| \quad (2.42)$$

and

$$\{\tilde{\lambda}/\tilde{\mu}\} = |a_{\lambda_i - \mu_j - i + j}| \quad (2.43)$$

The dimension of  $\{\lambda/\mu\}$  is defined by

$$f^{(\lambda/\mu)} = (w_{\lambda} - w_{\mu})! \left| \frac{1}{(\lambda_i - \mu_j - i + j)!} \right| \quad (2.44)$$

The skew S-functions  $\{\lambda/\mu\}$  are symmetric functions and hence may be expressed in terms of S-functions  $\{\nu\}$  of weight  $w_{\lambda} - w_{\mu}$ . The procedure for obtaining the terms  $\{\nu\}$  for a given  $\{\lambda/\mu\}$  is given by expanding the determinant in (2.92) and expressing the product of  $h_r$  as S-functions using (2.34) and may be written as

$$\{\lambda/\mu\} = j_{\lambda\mu}^{\nu} \{\nu\} \quad (2.45)$$

In section 2.7 we will give another procedure for obtaining the coefficient  $j_{\lambda\mu}^{\nu}$  due to Wybourne.

Equation (2.45) may be dimensionally checked by noting that

$$f^{(\lambda/\mu)} = j_{\lambda\mu}^{\nu} f^{(\nu)} \quad (2.46)$$

It is worth noting that the skew S-functions satisfy the identities

$$(\{\lambda\} + \{\mu\})/\{\nu\} = \{\lambda/\nu\} + \{\mu/\nu\} \quad (2.47)$$

$$\{(\{\lambda/\mu\})/\nu\} = \{(\{\lambda/\nu\})/\mu\} = \{\lambda/\mu/\nu\} \quad (2.48)$$

$$\{\lambda/(\mu+\nu)\} = \{\lambda/\mu\} + \{\lambda/\nu\} \quad (2.49)$$

$$\{\lambda/\mu/\nu\} = \{\lambda/(\mu \cdot \nu)\} \quad (2.50)$$

The relation between multiplication of S-functions and skew S-functions is such that if

$$\{\mu \cdot \nu\} = g_{\mu\nu}^{\lambda} \{\lambda\} \quad (2.51)$$

then

$$\{\lambda/\mu\} = g_{\mu\nu}^{\lambda} \{\nu\} \quad (2.52)$$

As a consequence eqns (2.51) and (2.52) may be taken as the definition of the skew S-function.

## 2.7 PARTITIONS, FRAMES AND NUMBERINGS

In Chapter 1 we mentioned that every order partition could be expressed by Young diagrams, but it is useful to

consider diagrams that are more general than the Young diagrams of order partitions. To that end we introduce the concept of a frame of  $(\lambda)$  which we denote by  $F(\lambda)$ . If  $Z$  is the set of integers then a frame is any finite subset  $F(\lambda)$  of  $Z \times Z$  defined as

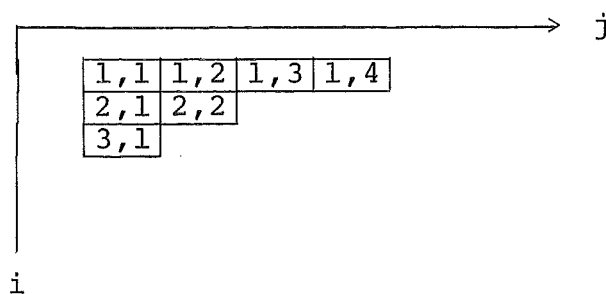
$$F(\lambda) = \{i, j(i)\} \quad (2.53)$$

where

$$i = 1, 2, \dots, p_\lambda$$

$$j(i) = 1, 2, \dots, \lambda_i$$

Thus a frame may be regarded as pattern of cells in the plane with the coordinates of a given cell being  $(i, j)$ . As an example we consider the partitions  $(421)$  of 7 and its frame is



If  $(\lambda)$  is an ordered partition the frame  $F(\lambda)$  will be said to be regular and will be equivalent to Young diagram.

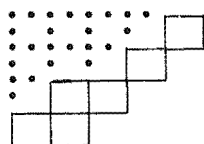
Now consider two partitions  $(\lambda) = (\lambda_1 \lambda_2 \dots \lambda_{p_\lambda})$  and  $(\mu) = (\mu_1 \mu_2 \dots \mu_{p_\mu})$  of weight  $w_\lambda$  and  $w_\mu$  respectively ( $w_\lambda > w_\mu$ ) such that  $p_\lambda > p_\mu$  and  $\lambda_i \geq \mu_i$ ,  $i = 1, 2, \dots, p_\mu$ . Let us superimpose the frame  $F(\mu)$  on that of  $F(\lambda)$  such that the upper left-hand corner coincides and  $F(\lambda) \supset F(\mu)$ . The cells of  $F(\lambda)$  not covered by  $F(\mu)$  will form a skew frame which

will be designated as  $F(\lambda/\mu)$ . If  $(\lambda)$  and  $(\lambda)$  are ordered partitions, then  $F(\lambda/\mu)$  will form a regular skew frame in the sense that no end cell in the  $i$ -th row appears to the left of the end cell of the  $j > i$  rows.

A skew frame may consist of several disconnected parts which are themselves skew frames. The association of a skew frame from  $F(\lambda/\mu)$  in terms of  $F(\lambda)$  and  $F(\mu)$  is by no means unique. Thus  $F(65^2 42/5^2 421)$  and  $F(64^2 32/5431)$  both yield the same skew frame



where the skew frame cells are bold and the empty cells of  $F(65421)$  and  $F(5431)$  are dotted in. empty rows and columns in  $F(\lambda/\mu)$  may be removed so as to bring disjoint pieces into contact. Thus the above skew frame may be replaced by the skew frame



corresponding to  $F(5432/431)$ .

Every skew frame  $F(\lambda/\mu)$  can be associated with a pair of minimal partitions  $(\lambda')$  and  $(\mu')$  as follow:

1. Draw the frame  $F(\lambda/\mu)$  and remove all empty rows and columns.

2. The remaining frame corresponds to  $F(\mu')$ .
3. The residual skew frame corresponds to  $F(\lambda'/\mu')$  with  $(\lambda')$  and  $(\mu')$  being the minimal partitions.

The skew frame associated with a pair of minimal partitions will be said to be a regular minimal skew frame  $F^m(\lambda/\mu)$ .

Thus the  $F(64^2 32/5431)$  and  $F(73^2 43/65421)$  have the same regular minimal skew frame  $F(5432/431)$  i.e.

$$\begin{aligned} F^m(65^2 42/5^2 421) &= F^m(64^2 32/5431) \\ &= F(5432/431). \end{aligned}$$

So far we have not endowed the frames with any numerical content. We now introduce the important concept of the numbering of a frame. Consider an arbitrary frame  $F_n$  having  $n$  cells. We define a numbering of  $F_n$  to be a map  $\eta: F_n \rightarrow \mathbb{Z}^+$  (where  $\mathbb{Z}^+$  is the set of positive integers) satisfying

$$\eta(i, j) \leq \eta(i', j') \quad \text{if } i = i' \text{ and } j < j' \quad (2.54a)$$

$$\eta(i, j) < \eta(i', j') \quad \text{if } j = j' \text{ and } i < i' \quad (2.54b)$$

This amounts to saying that the numbers in a given row must be weakly increasing (2.54a) and that in a given column must be strongly increasing (2.54b).

If  $F_n(\lambda)$  is a frame of a partition  $(\lambda)$  then numbering  $\eta$  of  $F_n(\lambda)$  produces a regular Young tableau while a numbering

of skew frame  $F_n(\lambda/\mu)$  with  $(\mu) \neq (0)$  produces a skew Young tableau.

Numberings of  $F_n$  may be constructed in various ways.

A standard numbering of  $F_n$  is a numbering  $\zeta$  of  $F_n$  such that  $\zeta$  is a one-to-one map of  $F_n$  onto the set  $\{1, 2, \dots, n\}$ . A standard numbering  $\zeta$  of the frame  $F_n(\lambda)$  of a partition  $(\lambda)$  yields a standard Young tableau and that of skew frame  $F_n(\lambda/\mu)$  a standard skew Young tableau.

The number of standard Young tableau associated with the standard numbering of the frame  $F_n(\lambda)$  is known from the theory of symmetric group  $S_n$  to be the dimension of the irrep  $[\lambda]$  of  $S_n$  or the dimension of  $\{\lambda\}$

$$f^{(\lambda)} = f^{[\lambda]} = \frac{n!}{H(\lambda)}$$

The number of standard skew Young tableaux associated with the standard numbering of the skew S-function  $\{\lambda/\mu\}$

$$f^{(\lambda/\mu)} = (w_\lambda - w_\mu)! \left| \frac{1}{(\lambda_i - \mu_j - i + j)!} \right|$$

For example the frame  $F(3^3 2/2^2)$  admits six standard numberings

1	1	1	2	2	3
2	3	4	3	4	4
34	24	33	14	14	12

the number is equal to

$$f(3^2 2 / 2^2) = 4! \begin{vmatrix} 1 & \frac{1}{2}! & \frac{1}{5}! \\ 1 & 1 & \frac{1}{4}! \\ 0 & 0 & \frac{1}{2}! \end{vmatrix} = 6$$

Another numbering is unitary numbering of an arbitrary frame  $F_n$  which is a map  $\xi$ , not necessarily one-to-one, from  $F_n$  into the set  $\{1, 2, \dots, N\}$ .

Thus if  $N = 2$ , there are just two unitary numberings of  $F_3(21)$

$$\begin{array}{cc} 1 & 1 \\ 2 & \end{array} \quad \text{and} \quad \begin{array}{cc} 1 & 2 \\ 2 & \end{array}$$

while for  $N = 3$  there are eight numberings of  $F_3(21)$

$$\begin{array}{cccccccccccccccc} 1 & 1 & & 1 & 2 & & 1 & 1 & & 1 & 2 & & 1 & 3 & & 1 & 3 & & 2 & 2 & & 2 & 3 \\ & & ' & & & ' & & & ' & & & ' & & & ' & & & ' & & & & ' & & \\ 2 & & & 2 & & & 3 & & & 3 & & & 2 & & & 3 & & & 3 & & & 3 & \end{array}$$

Unitary numberings also arise in skew frames  $F_n(\lambda/n)$ . In the case of  $F_4(32/1)$  we obtain for  $N = 3$  a set of twenty-one numberings.

For a frame  $F_n(\lambda)$  the number of distinct unitary numberings over  $\{1, 2, \dots, N\}$  is (Robinson, 1961).

$$D_N\{\lambda\} = G_N^{(\lambda)} / H(\lambda) \quad (2.55)$$

where  $H(\lambda)$  is the product of the hook lengths as given in (1.20) and

$$G_N^{(\lambda)} = \prod_{i,j} (N-i+j) \quad (2.56)$$

where  $i$  and  $j$  specify the positions of each cell in terms of the  $i$ -th row and  $j$ -th column respectively. Thus for the frame  $F_7(431)$  with  $N = 5$  we have

$$H(421) = 144$$

Labelling each cell of the frame  $F_7(421)$  with the numbers  $(i,j)$  we obtain

1,1	1,2	1,3	1,4
2,1	2,2		
3,1			

from which we deduce

$$\begin{aligned} G_5^{(421)} &= (5-1+1)(5-1+2)(5-1+3)(5-1+4) \\ &\quad (5-2+1)(5-2+2)(5-3+1) \\ &= 5.6.7.8.4.5.3 = 100800 \end{aligned}$$

leading via (2.65) to

$$D_5\{421\} = 700$$

But no such simple result appears to be known for  $D_N\{\lambda/\mu\}$ .

For every unitary numbering of  $F_n(\lambda)$  (or  $F_n(\lambda/\mu)$ ) if we read the entries in each row from right to left starting



from the uppermost row and get a sequence. Among them some may be lattice permutations which is a sequence  $a_1 a_2 \dots a_N$  in the symbols  $1, 2, \dots, N$  if for  $1 \leq r \leq N$  and  $1 \leq i \leq N-1$ , the number of occurrences of the symbol  $i$  in  $a_1 a_2 a_3 \dots a_r$  not less than the number of occurrences of  $i+1$ .

For  $F_n(\lambda)$  among the set of  $D_N\{\lambda\}$  unitary numberings there is only one numbering that corresponds to a lattice permutation. This unique lattice permutation is constructed by inserting the integer  $r$  in every cell of  $r$ -th row of the frame for  $r = 1, 2, \dots, p_\lambda$ . Thus for the frame  $F_n(4321^2)$  there is the numbering

```

1  1  1  1
2  2  2
3  3
4
5

```

which upon reading the entries in each row from right to left starting from the uppermost row yields the lattice permutation

```

1  1  1  1  2  2  2  3  3  4  5

```

But the set of unitary numberings of a skew frame  $F_n(\lambda/\mu)$  will usually involve a finite subset of numberings that correspond to lattice permutations.

For every lattice permutation there is a frame such that the number of cell in  $i$ -th row is equal to the number of  $r$ .

With this knowledge we now give the new procedure to reduce the  $\{\mu \cdot \nu\}$  and  $\{\lambda/\mu\}$  mentioned in secs. 2.5 and 2.6.

The procedure of  $\{\mu \cdot \nu\}$  is

1. give the unique unitary numbering corresponding lattice permutation for  $F_n(\mu)$ .
2. give a unitary numbering to the frame  $F_n(\nu)$  starting from right to left at the topmost row in such a manner as to produce a sequence of integers that, taken with those of  $F_n(\lambda)$ , produce a lattice permutation. The resulting lattice permutation may be associated with a frame  $F_{n+n'}(\lambda)$  by placing all ones in  $\lambda_1$  all the twos in  $\lambda_2$  etc.

3. repeat (2) and get all possible new frames  $F_{n+n'}(\lambda)$ , then

$$\{\mu \cdot \nu\} = g_{\mu\nu}^{\lambda} \{\lambda\}$$

4. use the (2.41) to check (3) such that we get all permissible  $\{\lambda\}$ .

In the case of  $\{321 \cdot 21\}$  we produce the lattice permutation numbering and their associated partitions  $(\lambda)$  as given below:

1 1 1    1 1    1 1    1 1  
 2 2    : 2    , 3    , 4    ,  
 3

(531) (52<sup>2</sup>) (521<sup>2</sup>)

1 2    1 3    1 2  
 3    , 2    , 4    ,  
 (432) (432) (431<sup>2</sup>)

$$\begin{array}{cccc}
1\ 4 & 1\ 2 & 1\ 3 & 1\ 4 \\
2 & , & 2 & , & 4 & , & 3 \\
(431^2) & (4^21) & (42^21) & (42^21) \\
\\ 
1\ 4 & 2\ 3 & 2\ 3 & 2\ 4 & 2\ 4 \\
5 & , & 3 & , & 4 & , & 3 & , & 5 \\
(421^3) & (3^3) & (3^221) & (3^221) & (3^21^3) \\
\\ 
3\ 4 & 3\ 4 \\
4 & , & 5 \\
(32^3) & (32^21^2)
\end{array}$$

from which we deduce

$$\begin{aligned}
\{321 \cdot 21\} &= \{531\} + \{52^2\} + \{521^2\} + 2\{432\} \\
&+ 2\{431^2\} + \{4^21\} + 2\{42^21\} + \{421^3\} \\
&+ \{3^3\} + 2\{3^221\} + \{3^21^3\} + \{32^3\} \\
&+ \{32^21^2\}
\end{aligned}$$

and (2.41) is satisfied

$$\begin{aligned}
f^{(321 \cdot 21)} &= \frac{9!}{6!3!} f^{(321)} f^{(21)} = 2688 \\
&= 162 + 120 + 189 + 2 \times 168 + 2 \times 216 + 84 \\
&+ 2 \times 216 + 189 + 42 + 2 \times 168 + 120 + 84 + 162 \\
&= 2688
\end{aligned}$$

The above procedure is none other than a restatement of the Littlewood-Richardson rule but simpler than it.

Using the procedure we get

$$\{1^s \cdot \rho\} = \sum_q \{1^{s+q}; \rho / 1^q\} \quad (2.57)$$

where for any partition  $\lambda$  into  $p$  non-vanishing  $p$  parts:

$$\{1^r; \lambda\} = \{\lambda_1+1, \lambda_2+1, \dots, \lambda_p+1, 1^{r-p}\} \quad \text{for } r \geq p \quad (1.58)$$

The Young diagram of  $(1^r; \lambda)$  is thus formed by adjoining the single column of length  $r$  corresponding to  $(1^r)$  to the left hand side of the Young diagram of  $(\lambda)$ . This same symbol is given a meaning in the case  $r < p$  by means of the modification rules (King, 1971)

$$\{1^r; \lambda\} = (-1)^x \{1^{p-1}; \lambda-h\} \quad \text{with } h = p-r-1$$

where  $p$  is the number of parts of  $\lambda$ , and  $h$  is the length of the continuous boundary strip or hook removed from the Young diagram of  $(\lambda)$  starting from the foot of the first, or left most, column and ending in the  $x$ -th column. The corresponding term vanishes unless the resulting Young diagram signified by  $(\lambda-h)$  is regular.

If we let the  $\{\rho\}$  of (2.67) be  $\{1^t\}$  then we get

$$\{1^s, 1^t\} = \sum_{n=0}^{\left\lfloor \frac{s+t}{2} \right\rfloor} \{2^n, 1^{s+t-2n}\} \quad (2.59)$$

For  $\{\lambda/\mu\}$  the procedure of reductions is

- (1) give the minimal skew frame  $F_n^m(\lambda/\mu)$  of  $F_n(\lambda/\mu)$
- (2) give the unitary numberings of the frame  $F_n^m(\lambda/\mu)$
- (3) choose such numberings which correspond to lattice permutations
- (4) give the frame  $F_n(v)$  which arises from the lattice permutations then

$$\{\lambda/\mu\} = j_{\lambda\mu}^{\nu}\{\nu\}$$

(5) use the (2.46) to check the result.

As an example we consider the  $\{76^2 542/6^3 31^2\}$ , the minimal skew frame  $F_n^m(76^2 542/6^3 31^2)$  is  $F_n(5431/42)$ . There are twelve distinct lattice permutations that arise in the unitary numbering ( $N \leq 4$ ), these are given below together with the partitions ( $\nu$ )

1	1	1	1
1 1	1 1	1 1	1 2
1 1 2	1 1 2	1 2 2	1 1 2
2	3	2	2
(52)	(51 <sup>2</sup> )	(43)	(43)
1	1	1	1
1 1	1 2	1 2	1 2
1 2 2	1 1 2	1 1 3	1 1 3
3	3	2	4
(421)	(421)	(421)	(41 <sup>2</sup> )
1	1	1	1
1 1	1 2	1 2	1 2
2 2 2	1 2 3	1 2 3	1 2 3
3	2	3	4
(3 <sup>2</sup> 1)	(3 <sup>2</sup> 1)	(32 <sup>2</sup> )	(321 <sup>2</sup> )

implying that

$$\begin{aligned} \{76^2 542/6^3 31^2\} &= \{52\} + \{51^2\} + 2\{43\} + 3\{4^2 1\} + \{41^3\} \\ &+ 2\{3^2 1\} + \{32^2\} + \{321^2\} \end{aligned}$$

and (2.46) is satisfied for

$$f(76^2 542 / 6^3 31^2) = 7! \begin{vmatrix} 1 & \frac{1}{4}! & \frac{1}{7}! & \frac{1}{8}! \\ 0 & \frac{1}{2}! & \frac{2}{5}! & \frac{1}{6}! \\ 0 & 1! & \frac{1}{3}! & \frac{1}{4}! \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= 7! / 18 = 280$$

and application of (2.27) to each of the partition  $\{v\}$  of the right of formula then gives

$$280 = 14 + 14 + 2 \times 14 + 3 \times 35 + 20 + 2 \times 21 + 21 + 35$$

## 2.8 S-FUNCTION SERIES

In much of our subsequent work we shall need to make extensive use of certain series of S-functions (Littlewood, 1950, MacDonald, 1979, Wybourne, 1970). The following series play a key role (King, 1975b).

$$A = \{0\} + \sum_{\alpha} (-1)^{w_{\alpha}/2} \{\alpha\} \quad B = \sum_{\beta} \{\beta\}$$

$$C = \{0\} + \sum_{\gamma} (-1)^{w_{\gamma}/2} \{\gamma\} \quad D = \sum_{\delta} \{\delta\}$$

$$E = \sum_{\xi} (-1)^{(w_{\xi}+r)/2} \{\xi\} \quad F = \sum_{\zeta} \{\zeta\}$$

$$G = \sum_{\xi} (-1)^{(w_{\xi}-r)/2} \{\xi\} \quad H = \sum_{\zeta} (-1)^{w_{\zeta}} \{\zeta\}$$

$$L = \sum_m (-1)^m \{1^m\}$$

$$M = \sum_m \{m\}$$

$$P = \sum_m (-1)^m \{m\}$$

$$Q = \sum_m \{1^m\} ,$$

where  $(\alpha)$  and  $(\gamma)$  are mutually associated partitions which in the Frobenius notation take the form

$$(\alpha) = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_1+1 & a_2+1 & \dots & a_r+1 \end{pmatrix}$$

$$(\gamma) = \begin{pmatrix} c_1+1 & c_2+1 & \dots & c_r+1 \\ c_1 & c_2 & \dots & c_r \end{pmatrix} ,$$

$(\delta)$  is a partition into even parts only and  $(\beta)$  is associated to  $(\delta)$ .  $(\zeta)$  is any partition and  $(\xi)$  is any self-associated partition with  $r$  being its rank.

For examples

$$A = \{0\} - \{1^2\} + \{21^2\} - \{31^3\} - \{2^3\} + \{41^4\} + \{32^21\} \dots$$

$$B = \{0\} + \{1^2\} + \{2^2\} + \{1^4\} + \{2^21^2\} + \{3^2\} + \{1^6\} + \{4^2\} \dots$$

$$C = \{0\} - \{2\} + \{31\} - \{41^2\} - \{3^2\} + \{51^3\} + \{431\} \dots$$

$$D = \{0\} + \{2\} + \{2^2\} + \{4\} + \{42\} + \{2^3\} + \{6\} + \dots$$

$$E = \{0\} - \{1\} + \{21\} - \{2^2\} - \{31^2\} + \{321\} + \{41^3\} \dots$$

$$F = \{0\} + \{1\} + \{2\} + \{1^2\} + \{3\} + \{21\} + \{1^3\} + \dots$$

$$G = \{0\} + \{1\} - \{21\} - \{2^2\} + \{31^2\} + \{321\} - \{41^3\} \dots$$

$$H = \{0\} - \{1\} + \{2\} + \{1^2\} - \{3\} - \{21\} - \{1^3\} \dots$$

$$L = \{0\} - \{1\} + \{1^2\} - \{1^3\} + \{1^4\} - \{1^4\} + \{1^6\} \dots$$

$$M = \{0\} + \{1\} + \{2\} + \{3\} + \{4\} + \{5\} + \{6\} \dots$$

$$P = \{0\} - \{1\} + \{2\} - \{3\} + \{4\} - \{5\} + \{6\} \dots$$

$$Q = \{0\} + \{1\} + \{1^2\} + \{1^3\} + \{1^4\} + \{1^5\} + \{1^6\} \dots$$

These series occur as mutually inverse pairs

$$AB = CD = EF = GH = LM = PQ = \{0\} = 1 \quad (2.60)$$

Also

$$LA = PC = E$$

$$MB = QD = F$$

$$MC = AQ = G$$

$$LD = PB = H \quad (2.61)$$

In addition to the above series we shall make use of both.

$$R = \{0\} - 2 \sum_{a,b} (-1)^{a+b+1} \begin{pmatrix} a \\ b \end{pmatrix} \quad S = \{0\} + 2 \sum_{a,b} \begin{pmatrix} a \\ b \end{pmatrix}$$

where the Frobenius notation has been used once again, and

$$V = \sum_{\omega} (-1)^q \{\tilde{\omega}\}$$

$$W = \sum_{\omega} (-1)^q \{\omega\}$$

$$X = \sum_{\omega} \{\tilde{\omega}\}$$

$$Y = \sum_{\omega} \{\omega\}$$

where  $(\omega)$  is a partition of an even number into at most two parts, the second of which is  $q$ .

We also have the examples

$$R = \{0\} + 2\{1\} - 2\{2\} - 2\{1^2\} + 2\{21\} + 2\{3\} + 2\{1^3\} \dots$$



$$S = \{0\} + 2\{1\} + 2\{2\} + 2\{1^2\} + 2\{21\} + 2\{3\} + 2\{1^3\} \dots$$

$$V = \{0\} + \{1^2\} + \{1^4\} + \{2^2\} - \{2\} - \{21^2\} - \{21^4\} \dots$$

$$W = \{0\} + \{2\} + \{4\} + \{2^2\} - \{1^2\} - \{31\} - \{51\} \dots$$

$$X = \{0\} + \{1^2\} + \{2\} + \{1^4\} + \{21^2\} + \{2^2\} + \dots$$

$$Y = \{0\} + \{2\} + \{1^2\} + \{4\} + \{31\} + \{2^2\} + \dots$$

We readily find that

$$RS = VW = \{0\} = 1 \quad (2.62)$$

and

$$PM = AD = W \quad LQ = BC = V$$

$$MQ = FG = S \quad LP = HE = R \quad (2.63)$$

We shall frequently form series of skew S-functions making use of abbreviations such as

$$\{\lambda/A\} = \sum_{\alpha} (-1)^{w_{\alpha}/2} \{\lambda/\alpha\}$$

and

$$\{\lambda/\beta\} = \sum_{\beta} \{\lambda/\beta\}$$

etc.

The following identities involving skew S-functions and the outer product of S-functions also prove useful

$$\{(\sigma \cdot \tau)/Z\} = \{\sigma/Z\} \cdot \{\tau/Z\} \quad (2.64)$$

for  $Z = L, M, P, Q, R, S, V$  and  $W$

$$\{ \{ \sigma \cdot \tau \} / Z \} = \sum_{\zeta} \{ \sigma / \zeta Z \} \cdot \{ \tau / \zeta Z \} \quad (2.65)$$

for  $Z = B, D, F$  and  $H$ .

and

$$\{ \{ \sigma \cdot \tau \} / Z \} = \sum_{\zeta} (-1)^{w_{\zeta}} \{ \sigma / \zeta Z \} \cdot \{ \tau / \zeta Z \} \quad (2.66)$$

for  $Z = A, C, E$  and  $G$ .

## 2.9 HALL-LITTLEWOOD FUNCTIONS AND Q-FUNCTIONS

The  $S$ -functions were introduced by Schur in his development of the theory of the ordinary irreps of  $S_n$  (Schur, 1901). Later, in his study of the spin irreps of  $S_n$ , he introduced a second symmetric function termed the  $Q$ -function (Schur, 1911). It was much later realized that the  $S$ - and  $Q$ -functions were particular cases of what are now known as Hall-Littlewood functions (Hall, 1957, Littlewood, 1961). The properties of the Hall-Littlewood functions have been surveyed by Morris (1976) and a concise description has been given by Thomas (1976).

The Schur functions may be generalized by consideration of the expressions

$$\prod \frac{(1 - t \alpha_i x)}{(1 - s \alpha_i x)} = \sum_{r=0}^{\infty} q_r x^r \quad (2.67)$$

with

$$\begin{aligned} \{ \lambda \}_q &= |q_{\lambda_i - i + j}| \\ &= \prod_{i < j} (1 - R_{ij}) q(\lambda) \end{aligned} \quad (2.68a)$$

where

$$q_{(\lambda)} = q_{\lambda_1} q_{\lambda_2} \cdots \quad (2.68b)$$

and  $R_{ij}$  is a raising operator as defined in Sec. 2.4. It is readily seen that  $t = 0$ ,  $s = 1$  yields the usual S-functions.

A further generalization is made possible by considering two sets of indeterminants  $\alpha_1, \alpha_2, \cdots, \alpha_n$  and  $\beta_1, \beta_2, \cdots, \beta_n$  and the function

$$\prod_{i,j} \frac{(1-t\alpha_i\beta_jx)}{(1-\alpha_i\beta_jx)} = \sum_{r=0}^{\infty} p_r x^r \quad (2.69)$$

together with the requirement that  $P_r$  is expressed in the form

$$P_r = \sum_{(\lambda)} K_{(\lambda)}(t) Q_{(\lambda)}(t) Q'_{(\lambda)}(t) \quad (2.70)$$

where  $Q_{(\lambda)}(t)$  is a symmetric function in  $\alpha_1, \alpha_2, \cdots, \alpha_n$ ,  $Q'_{(\lambda)}(t)$  is the same symmetric function <sup>in</sup>  $\beta_1, \cdots, \beta_n$  and  $K_{(\lambda)}(t)$  is a polynomial in  $t$  which depends on the partition  $(\lambda)$ . The functions  $Q_{(\lambda)}(t)$  are known as Hall-Littlewood functions.

The Young raising operators  $R_{ij}$  may be used to express the generalized S-functions  $\{\lambda\}_q$  of (2.68) in terms of Hall-Littlewood function to give (Littlewood, 1961, Thomas 1976).

$$\{\lambda\}_q = \prod_{i < j} (1-tR_{ij}) Q_{(\lambda)}(t) \quad (2.71a)$$

and vice versa

$$\begin{aligned}
Q_{(\lambda)}(t) &= \prod_{i < j} \frac{1}{(1 - tR_{ij})} \{\lambda\}_q \\
&= \prod_{i < j} (1 + tR_{ij} + t^2 R_{ij}^2 + \dots) \{\lambda\}_q
\end{aligned} \tag{2.71b}$$

The Hall-Littlewood functions  $Q_{(\lambda)}(-1)$  play a similar role for spin irreps of  $S_n$  as do the S-function for ordinary irreps of  $S_n$  (cf. next chapter). Henceforth, we shall write

$$Q_{(\lambda)} = Q_{(\lambda)}(-1)$$

and refer to the  $Q_{(\lambda)}$  simply as Q-functions introduced by Schur (1911). Each Q-function will be associated with a partition  $(\lambda)$  of  $n$ . The partition need not be in standard form. Q-functions corresponding to non-standard partitions may be converted into standard descending order by noting the following four rules:

1. If any two parts are equal the Q-function is zero.

$$2. \quad Q(\dots \lambda_i, \lambda_{i+1}, \dots) = -Q(\dots \lambda_{i+1}, \lambda_i, \dots) \tag{2.72}$$

3. A Q-function will be zero if any part is negative and the magnitude of every part is different.

$$4. \quad Q(\dots \lambda_i, -\lambda_i, \dots) = 0 \tag{2.73a}$$

while

$$Q(\dots -\lambda_i, \lambda_i, \dots) = 2(-1)^{\lambda_i} Q(\dots \lambda_{i-1}, \lambda_{i+1}, \dots) \tag{2.73b}$$

Application of the above rules allows us to reduce any Q-functions either to zero or to the form  $Q_{(\lambda)}$  where  $(\lambda)$  is a partition of  $n$  into  $k$  unequal parts such that

$$\lambda_1 > \lambda_2 > \cdots \lambda_k > 0$$

(2.74)

The identical situation holds for the existence of spin irreps of  $S_n$ .

### CHAPTER 3

#### SYMMETRIC FUNCTIONS AND THE REPRESENTATIONS OF GROUPS

##### 3.1 IMMANANTS OF MATRICES

Symmetric group  $S_n$  has two one-dimension irreps  $[n]$  and  $[1^n]$ . With use of the character  $\chi^{[n]}$  and  $\chi^{[1^n]}$  of these irreps the determinant and permanent of a  $n \times n$  matrix  $(a_{st})$  can be written as

$$|a_{st}| = \sum \chi^{[1^n]}(q) P_q \quad (3.1)$$

and

$$|a_{st}|^+ = \sum \chi^{[n]}(q) P_q \quad (3.2)$$

where the summation is over the  $n!$  permutations of the symmetric group  $S_n$  and  $q$  denotes any permutations  $e_1, e_2, \dots, e_n$

$$P_q = a_{1e_1} a_{2e_2} \dots a_{ne_n} \quad (3.3)$$

The concept of the immanant of a matrix is a natural extension of the idea of the determinant and permanent of a matrix (Littlewood et al 1934). The immanant of a  $n \times n$  matrix  $(a_{st})$  is defined as

$$|a_{st}|^{(\lambda)} = \sum \chi^{[\lambda]}(q) P_q \quad (3.4)$$

where  $\chi^{[\lambda]}$  is the character of  $S_n$  corresponding to the irrep  $[\lambda]$  and summation is over the  $n!$  permutations of  $S_n$ .

Consider the case of the immanants of a  $3 \times 3$  matrix  $(a_{st})$ . We have the character table for  $S_3$  as

class	$(1^3)$	$(21)$	$(3)$
order	1	3	2
$[3]$	1	1	1
$[21]$	2	0	-1
$[1^3]$	1	-1	1

hence the three immanants

$$|a_{st}|^{(3)} = a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} \\ + a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{31}$$

$$|a_{st}|^{(21)} = 2a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}$$

$$|a_{st}|^{(1^3)} = a_{11}a_{12}a_{13} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ - a_{13}a_{23}a_{31} + a_{12}a_{23}a_{21} + a_{31}a_{21}a_{32}$$

It is easily seen that  $|a_{st}|^{(3)}$  is simply the permanent and  $|a_{st}|^{(1^3)}$  the determinant of the matrix  $(a_{st})$  and generally

$$|a_{st}|^+ = |a_{st}|^{(n)} \quad (3.5)$$

and

$$|a_{st}| = |a_{st}|^{(1^n)} \quad (3.6)$$

### 3.2 SCHUR FUNCTIONS AND SYMMETRIC GROUPS

If we represent by  $(z_r)$  the matrix

$$(z_r) = \begin{pmatrix} s_1 & 1 & & & \\ & s_2 & s_1 & 2 & \\ & s_3 & s_2 & s_1 & 3 \\ & \dots & \dots & \dots & \dots \\ s_r & s_{r-1} & \dots & \dots & s_1 \end{pmatrix} \quad (3.7)$$

then from (2.21) and (2.23) we have the results

$$r!a_r = |z_r| \quad (3.8)$$

and

$$r!h_r = |z_r|^+ \quad (3.9)$$

If  $(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_p)$  is a partition of  $r$ . We define a function  $\{\lambda\} \equiv \{\lambda_1, \lambda_2, \dots, \lambda_p\}$  by writing by analogy with (3.8) and (3.9)

$$r!\{\lambda\} = |\lambda_r|^{(\lambda)} \quad (3.10)$$

clearly

$$\{r\} = h_r \quad (3.11)$$



and

$$\{1^r\} = a_r \quad (3.12)$$

Not surprisingly  $\{\lambda\}$  will shortly be identified as the Schur function mentioned in the previous chapter.

Let  $\rho$  denote the class  $(\dots 2^{v_2} 1^{v_1})$  of order  $g_\rho$  of  $s_r$  and let  $s_{(\rho)} = s^{v_1} s^{v_2} \dots$  then we have (Littlewood 1950)

$$r! \{\lambda\} = \sum_{(\rho)} \chi_{(\rho)}^{[\lambda]} g_\rho s_{(\rho)} \quad (3.13)$$

and conversely

$$s_{(\rho)} = \sum_{(\lambda)} \chi_{(\rho)}^{[\lambda]} \{\lambda\} \quad (3.14)$$

Thus for  $S_3$  we have

$$6h_3 = 6\{3\} = s_1^3 + 3s_1s_2 + 3s_3$$

$$6\{21\} = 2s_1^3 - 2s_3$$

$$6a_3 = 6\{1^3\} = s_1^3 - 3s_1s_2 + 2s_3$$

and

$$s_1^3 = \{3\} + 2\{21\} + \{1^3\}$$

$$s_1s_2 = \{3\} - \{1^3\}$$

$$s_3 = \{3\} - \{21\} + \{1^3\}$$

(3.13) and (3.14) provide a crucial link between S-functions and the characters of the symmetric group. (3.14)

shows clearly that the characters are simply the elements of a matrix that relates the S-functions to the symmetric functions  $s_{(\rho)}$ .

The celebrated formula for Frobenius leads directly to the identification of the definition of  $\{\lambda\}$  (3.10) with that defined, in (2.24), in terms of determinantal expansions involving  $(x)$ .

The Frobenius' formula for the character of  $S_n$  is

$$\begin{aligned} S_{(\rho)} \Delta(x_1, \dots, x_n) \\ = \sum_{(\lambda)} \chi_{(\rho)}^{[\lambda]} \sum_p \delta_p x_1^{\lambda_1+n-1} x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \dots x_n^{\lambda_n} \end{aligned} \quad (3.15)$$

where

$$\begin{aligned} \Delta(x_1, \dots, x_n) &= \prod_{i < j} (x_i - x_j) \\ &= \sum_p \delta_p x_1^{n-1} x_2^{n-2} \dots x_{n-1} x_n^0 \end{aligned} \quad (3.16)$$

and  $p$  is any permutation of the variable  $x_i$  and  $\delta_p = \pm 1$ , depending on whether  $p$  is even or odd permutation.

From (3.15) we have

$$\begin{aligned} \sum_{(\rho)} g_{\rho} \chi_{(\rho)}^{[\lambda]} s_{(\rho)} \Delta(x_1, \dots, x_n) \\ = \sum_{(\rho), (\mu)} \chi_{(\rho)}^{[\lambda]} \chi_{(\rho)}^{[\mu]} \sum_p \delta_p x_1^{\mu_1+n-1} x_2^{\mu_2+n-2} \dots x_n^{\mu_n} \\ = r! \sum_p \delta_p p \prod_i x_i^{\lambda_i+n-i} \end{aligned}$$

and hence

$$\begin{aligned} \{\lambda\} &= \frac{1}{r!} \sum_{\rho} g_{\rho} \chi_{(\rho)}^{[\lambda]} s_{(\rho)} = \frac{\sum_{\mathbf{p}} \delta_{\mathbf{p}} p_i^{\lambda_i + n - i}}{\sum_{\mathbf{p}} \delta_{\mathbf{p}} p_i^{n - i}} \\ &= \frac{|x_j^{\lambda_i + n - i}|}{|x_j^{n - i}|} \end{aligned}$$

### 3.3 SCHUR FUNCTIONS AND UNITARY GROUP

The connection between S-functions and the classical Lie groups comes about through the fact that a S-function  $\{\lambda\}$  of the characteristic roots of a matrix  $A$  is the trace of an invariant matrix  $A^{\{\lambda\}}$ , the tensor rep of given symmetry type  $\{\lambda\}$ , whose elements are polynomials in the elements of  $A$ . If the matrix  $A$  is an element of a group  $G$  the map of  $A \rightarrow A^{\{\lambda\}}$  gives a rep of that group  $G$  whose character is the S-function  $\{\lambda\}$ . The Kronecker product decomposition for the reps is equivalent to the outer product of S-functions.

In the case of unitary group  $U_N$  the rep of the  $N \times N$  matrices  $A$  by the matrices  $A^{\{\lambda\}}$  is irreducible and as a result we shall use the partitions  $\{\lambda\}$  that are associated with the S-functions to label the irreps of  $U_N$ .

The S-function  $\{\lambda\}$  will be null if  $p_{\lambda} > N$  and hence

$$\chi^{\{\lambda\}} = 0 \quad \text{if} \quad p_{\lambda} > N$$

The dimension  $D_N\{\lambda\}$  of irrep  $\{\lambda\}$  of  $U_N$  is equal to the number of distinct unitary numberings of  $F_n(\lambda)$  given

in Chapter 2.

$$D_N\{\lambda\} = G_N^{(\lambda)} / H(\lambda)$$

### 3.4 THE OUTER PRODUCT OF IRREPS OF SYMMETRIC GROUP

We can consider the direct product group  $S_n \times S_m$  as a subgroup of  $S_{n+m}$ . The irreps of  $S_n \times S_m$  are of the form  $[\mu][\nu]$ , where  $[\mu], [\nu]$  are irreps of  $S_n, S_m$  respectively. The rep of  $S_{n+m}$  induced from  $[\mu][\nu]$  is called the outer product of  $[\mu]$  and  $[\nu]$  and denoted by  $[\mu] \boxtimes [\nu]$ . Generally the rep  $[\mu] \boxtimes [\nu]$  of  $S_{n+m}$  is reducible. Due to the dual property of  $U_N$  and  $S_n$ : (Miller, 1972). If

$$\{\mu \cdot \nu\} = g_{\mu\nu}^{\lambda} \{\lambda\}$$

for  $U_N$ , where  $(\mu)$  is the partition of  $m$  and  $(\nu)$  the partition of  $n$  then

$$[\mu] \boxtimes [\nu] = g_{\mu\nu}^{\lambda} [\lambda] \quad (3.17)$$

for  $S_{n+m}$ . So the evaluation of the  $[\mu] \boxtimes [\nu]$  is equivalent to the evaluation of the outer product of S-function given in sec. 2.7, chapter 2 while the skew S-function  $\{\lambda/1\}$  corresponds to the branching rule of  $S_n \downarrow S_{n-1}$ . i.e.

$$\text{if } \{\lambda/1\} = j_{1\lambda}^{\nu} \{\nu\}$$

then we have

$$S_n \downarrow S_{n-1}$$

$$[\lambda] \downarrow j_{1\lambda}^{\nu} [\nu] \quad (3.18)$$

where  $[\lambda]$  is irreps of  $S_n$ ,  $[\nu]$  are the irreps of  $S_{n-1}$ .

### 3.5 THE INNER PRODUCT OF S-FUNCTIONS

The reduction of the Kronecker product  $[\lambda][\mu]$  for  $S_n$  is considerably more complex than for the outer product. Littlewood (1956) had attempted to solve the problem by defining an inner product of S-functions as follow: If  $[\mu]$ ,  $[\nu]$  are the irreps of  $S_n$  and

$$\chi^{[\mu]} \chi^{[\nu]} = k_{\mu\nu}^{\lambda} \chi^{[\lambda]} \quad (3.19a)$$

then

$$\{\mu\} \circ \{\nu\} = k_{\mu\nu}^{\lambda} \{\lambda\} \quad (3.19b)$$

is called the inner product of the S-functions  $\{\mu\}$  and  $\{\nu\}$ . The problem of expressing the Kronecker product of two irreps of  $S_n$  as a direct sum of  $S_n$  is then equivalent to evaluating the inner product  $\{\lambda\} \circ \{\mu\}$ .

It follows from (3.19b) that the dimension of  $\{\mu\} \circ \{\nu\}$  is

$$f^{\{\mu\} \circ \{\nu\}} = f^{[\mu]} \cdot f^{[\nu]} = k_{\mu\nu}^{\lambda} f^{[\lambda]} \quad (3.20)$$

Littlewood (1956) had developed a method of evaluating S-function inner products where character tables are not

required. His key theorem may be written as

$$(\{\lambda \cdot \mu\} \circ \{\nu\}) = (\{\lambda\} \circ \{\nu/\sigma\}) \cdot (\{\mu\} \circ \{\sigma\}) \quad (3.21)$$

In using the above result we make use of

$$\{\lambda\} \circ h_n = h_n \circ \{\lambda\} = \{\lambda\} \quad (3.22)$$

To illustrate the application of (3.21) consider the evaluation of  $\{51\} \circ \{321\}$ . First expand  $\{51\}$  into products of S-function involving just one part using the method of raising operator to give

$$\{51\} = \{5\} \cdot \{1\} - \{6\}$$

Use of (3.20) then gives

$$\{5 \cdot 1\} \circ \{321\} = (\{5\} \circ \{321/\sigma\}) \cdot (\{1\} \circ \{\sigma\})$$

but here  $\{\sigma\} = \{1\}$  and hence

$$\begin{aligned} \{5 \cdot 1\} \circ \{321\} &= \{321/1\} \cdot \{1\} \\ &= (\{32\} + \{31^2\} + \{2^2 1\}) \cdot \{1\} \\ &= \{42\} + \{41^2\} + \{3^2\} + 3\{321\} + \{31^3\} + \{2^2\} \\ &\quad + \{2^2 1^2\} \end{aligned}$$

but

$$(\{5 \cdot 1\}) \circ \{321\} = \{51\} \circ \{321\} + \{6\} \circ \{321\}$$

$$\therefore \{51\} \circ \{321\} = \{5 \cdot 1\} \circ \{321\} - \{321\}$$

$$= \{42\} + \{41^2\} + \{3^2\} + 2\{321\} + \{31^3\} + \{2^3\} + \{2^2 1^2\}$$

We may check our result by (3.19)

$$5 \times 16 = 80 = 9 + 10 + 5 + 2 \times 16 + 10 + 5 + 9 = 80$$

A number of special formulas exist. In particular

$$\{n-1,1\} \circ \{n-1,1\} = \{n\} + \{n-1,1\} + \{n-2,2\} + \{n-2,1^2\} \quad (3.33a)$$

$$\{n-1,1\} \circ \{n-2,2\} = \{n-1,1\} + \{n-2,2\} + \{n-2,1^2\}$$

$$+ \{n-3,3\} + \{n-3,2,1\} \quad (3.23b)$$

$$\{n-1,1\} \circ \{n-2,1^2\} = \{n-1,1\} + \{n-2,2\} + \{n-2,1^2\}$$

$$+ \{n-3,2,1\} + \{n-3,1^3\} \quad (3.23c)$$

$$\{n-2,2\} \circ \{n-2,2\} = \{n\} + \{n-1,1\} + 2\{n-2,2\} + \{n-2,1^2\}$$

$$+ \{n-2,3\} + 2\{n-3,2,1\} + \{n-2,1^3\}$$

$$+ \{n-4,4\} + \{n-4,3,1\} + \{n-4,2^2\} \quad (3.23d)$$

The above formulas sometimes yield S-functions with parts that are not in standard order. These S-functions may

be transformed into standard form by the rules (2.31) mentioned in sec. 2.3 chapter 2. The above formulas suggest the idea of a reduced notation for the ordinary rep of  $S_n$  which we shall discuss in chapter 5.

### 3.6 Q-FUNCTIONS AND SPIN IRREPS OF SYMMETRIC GROUP

The connection between Q-functions and the spin characters of  $S_n$  is made explicit by Schur's relation (Schur 1911)

$$Q_{(\lambda)} = 2^{(k+p+\epsilon)/2} \sum_{(\pi)} \frac{g_{\pi}}{2g} \chi_{(\pi)}^{[\lambda]'} s_{(\pi)} \quad (3.24)$$

where  $\chi_{(\pi)}^{[\lambda]'}$  is a simple spin character of the class  $(\pi) = (\dots 3^{\alpha_3} 1^{\alpha_1})$  involving odd cycles only,  $k$  is the parts of partition of  $(\lambda)$ ,  $p = \alpha_1 + \alpha_3 + \dots$ ,  $g_{\pi}$  is the order of the class  $(\pi)$ ,  $g$  the order of  $S_n$ ,  $s_{(\pi)} = s_1^{\alpha_1} s_3^{\alpha_3} \dots$  and  $\epsilon = 0$  or  $1$  accordingly as  $(n-k)$  is even or odd.

### 3.7 OUTER PRODUCT OF Q-FUNCTIONS

The outer product of two Q-functions, say  $Q_{(\lambda)}$  and  $Q_{(\nu)}$ , may be resolved into a sum of Q-functions of weight

$w_{\lambda} = w_{\mu} + w_{\nu}$  to give

$$Q_{(\mu)} \cdot Q_{(\nu)} = a_{\mu\nu}^{\lambda} Q_{(\lambda)} \quad (3.25)$$

The non-negative coefficients  $a_{\mu\nu}^{\lambda}$  may be determined by use



of (2.81b) to expand each  $Q$ -function as a sum of  $S$ -functions  $\{\lambda\}_q$ , then the outer products of the  $S$ -functions evaluated by the Littlewood-Richardson rule, as in sec 2.7, and then the resulting  $S$ -functions converted back into  $Q$ -functions using (2.81a). Outer  $Q$ -function products are closely related to the induced rep of  $S_{n+m}$  from  $S_n \times S_m$ .

The evaluation of (3.25) may be dimensionally verified noting that

$$\begin{aligned}
 & 2^{(p_\mu + p_\nu + \epsilon_\mu + \epsilon_\nu)/2} \frac{(w_\mu + w_\nu)!}{w_\mu! w_\nu!} f^{[\mu], \dagger} f^{[\mu], \dagger} \\
 &= \sum_{\lambda} 2^{(p_\lambda + \epsilon_\lambda)/2} a_{\mu\nu}^{\lambda} f^{[\lambda], \dagger}
 \end{aligned} \tag{3.26}$$

Thus to evaluate  $Q_{(2)} \cdot Q_{(21)}$  we first use (2.81b) with  $t = -1$  to give

$$Q_{(2)} = \{2\}_q$$

$$Q_{(21)} = \{21\}_q - \{3\}_q$$

Forming the outer products of the  $S$ -functions gives

$$\{2\}_q \cdot (\{21\}_q - \{3\}_q) = \{2^2 1\}_q + \{31^2\}_q - \{5\}_q$$

Application of (2.81a) then gives

$$\begin{aligned}
 \{2^2 1\}_q &= (1+R_{12})(1+R_{13})(1+R_{23})Q_{(2^2 1)} \\
 &= (1+R_{12})(1+R_{13})(Q_{(2^2 1)} + Q_{(23)})
 \end{aligned}$$

$$\begin{aligned}
&= (1+R_{12}) (Q_{(2^2 1)} + Q_{(23)} + Q_{(32)} + Q_{(23-1)}) \\
&= Q_{(2^2 1)} + Q_{(23)} + Q_{(32)} + Q_{(23-1)} + Q_{(31^2)} \\
&\quad + Q_{(32)} + Q_{(41)} + Q_{(32-1)} \\
&= Q_{(32)} + Q_{(41)}
\end{aligned}$$

where in the last line we have used the standardizing rules given in ith sec. 2.9. Likewise

$$\begin{aligned}
\{31^2\}_q &= Q_{(32)} + Q_{(41)} + Q_{(5)} \\
\{5\}_q &= Q_{(5)}
\end{aligned}$$

and hence

$$Q_{(2)} \cdot Q_{(21)} = 2Q_{(32)} + 2Q_{(41)}$$

For later use we note the general result (Morris 1962)

$$\begin{aligned}
Q_{(\lambda)} \cdot Q_{(1)} &= 2[Q_{(\lambda_1+1, \dots, \lambda_k)} + Q_{(\lambda_1, \dots, \lambda_k+1)} \\
&\quad + Q_{(\lambda_1, \lambda_2, \dots, \lambda_k, 1)}]
\end{aligned} \tag{3.27}$$

Skew  $Q$ -functions may be defined by writing

$$Q_{(\lambda/\mu)} = b_{\mu\nu}^{\lambda} Q_{(\nu)} \tag{3.28}$$

Where  $a_{\mu\nu}^{\lambda}$  is the same as the coefficient that appears in the outer product

$$Q_{(\mu)} Q_{(\nu)} = a_{\mu\nu}^{\lambda} Q_{(\lambda)}$$

Thus noting (3.26) we have

$$\begin{aligned} Q_{(\lambda/1)} &= 2[Q_{(\lambda_1-1, \dots, \lambda_k)} + \dots + Q_{(\lambda_1, \dots, \lambda_k-1)}] \\ &\quad - \delta_{\lambda,1} Q_{(\lambda_1, \dots, \lambda_k-1)} \end{aligned} \quad (3.29)$$

Just like skew S-functions, the skew Q-function  $Q_{(\lambda/1)}$  corresponds to the branching rule  $S_n \downarrow S_{n-1}$  for spin reps.

### 3.8 INNER PRODUCT OF Q- AND S-FUNCTIONS

It is useful to define an inner product of a Q-function with an S-function defined on the same indeterminants by writing

$$Q_{(\mu)} \circ \{\nu\} = b_{\mu\nu}^{\lambda} Q_{(\lambda)} \quad (3.30)$$

where  $(\lambda), (\mu), (\nu) \rightarrow n$ . The inner product may be evaluated by using (2.81b) to express  $Q_{(\lambda)}$  as a sum of S-functions  $\{\lambda\}_q$ , then evaluating the S-function inner products and finally converting the resultant S-functions back into Q-functions using (2.81a).

In the case of  $(\lambda) = (n)$  equation (3.30) becomes

$$\begin{aligned} Q_{(n)} \circ \{\mu\}_q &= \{\mu\}_q = \prod_{i>j} (1+R_{ij}) Q_{(\mu)} \\ &= b_{n\mu}^{\lambda} Q_{(\lambda)} \end{aligned} \quad (3.31)$$

(3.31) may be checked by noting that

$$2^{[n-n(\bmod 2)/2]} f[\mu] = 2^{[(p_\mu-n(\bmod 2)/2]} b_{n\mu}^\lambda f[\lambda] \quad (3.32)$$

### 3.9 THE PLETHYSM OF S-FUNCTIONS

Consider the rep  $\{1\}$  of  $U_N$ , natural rep of  $U_N$ , the powers of degree  $r$  of  $\{1\}$  form the reducible rep of  $U_N$  and the different irreps of  $\{\lambda\}$  of  $U_N$  are contained in the decomposition of the  $r$ th Kronecker product

$$\{1\}^r = \sum f^{[\lambda]} \{\lambda\} \quad (3.33)$$

where the  $(\lambda)$ 's are partitions of  $r$  and  $f^{[\lambda]}$  is the number of times a given irrep  $\{\lambda\}$  occurs in the decomposition. It follows from the Weyl reciprocity theorem of  $U_N$  and  $S_r$  that  $f^{[\lambda]}$  is just the dimension of the irrep  $[\lambda]$  of the symmetric  $S_r$ .

If  $\{\mu\}$  is the irrep of  $U_M$  and  $D_M\{\mu\} = N$  then

$$\{\mu\}^r = \sum f^\lambda(\{\mu\}) \{\lambda\} \quad (3.34)$$

where  $w_\lambda = r, f^\lambda$  is the dimension of irrep  $[\lambda]$  of  $S_r$ .  $(\{\lambda\})^{\{\lambda\}}$  is regarded as a subset of the symmetry type irrep  $\{\lambda\}$  of  $U_N$  and is also a rep of  $U_M$ . Generally it is reducible for  $U_M$  and may be written as

$$(\{\mu\})^{\{\lambda\}} = c_{\mu\lambda}^v \{v\} \quad (3.35)$$

where  $\{\nu\}$  is the symmetry type irrep of  $U_M$ .

Equation (3.35) may be taken as defining the plethysm of S-functions

$$\{\mu\} \otimes \{\lambda\} = C_{\mu\lambda}^{\nu} \{\nu\} \quad (3.36)$$

where the symbol  $(\otimes)$  is used to indicate the operation of plethysm and  $\{\mu\} \otimes \{\lambda\}$  is read as " $\{\mu\}$  plethysm  $\{\lambda\}$ ".

The operation of plethysm is associative and distributive on the right with respect to addition, subtraction, and multiplication. It is assumed that the operation  $\otimes$  precedes ordinary S-function multiplication. We may readily deduce that

$$A \otimes (B \cdot C) = A \otimes B \cdot A \otimes C \quad (3.37)$$

$$A \otimes (B \pm C) = A \otimes B \pm A \otimes C \quad (3.38)$$

$$A \otimes (B) \otimes C = A \otimes (B \otimes C) \quad (3.39)$$

The operation is not distributive with respect to addition or subtraction or multiplication on the left. Littlewood (1950) has derived the additional rules that complete the definition of the algebra:

$$(A+B) \otimes \{\lambda\} = \sum A \otimes \{\lambda/\mu\} \cdot B \otimes \{\mu\} \quad (3.40)$$

$$(A-B) \otimes \{\lambda\} = \sum (-1)^{w_{\mu}} A \otimes \{\lambda/\mu\} \cdot B \otimes \{\tilde{\mu}\} \quad (3.41)$$

$$(A \cdot B) \otimes \{\lambda\} = \sum A \otimes (\{\lambda\} \circ \{\mu\}) \cdot B \otimes \{\mu\} \quad (3.42)$$

$$\begin{aligned}
 \{\lambda\} \otimes \{\mu\} &= \{\tilde{\lambda}\} \otimes \{\mu\} & w_{\mu} \text{ even} \\
 &= \{\tilde{\lambda}\} \otimes \{\tilde{\mu}\} & w_{\mu} \text{ odd}
 \end{aligned}
 \tag{3.43}$$

where the summations are over all compatible  $(\mu)$ . We note that the symbols on the left of  $\otimes$  need not be S-functions but can be any suitable functions defined on the variables.

The separation of the plethysm

$$\{\mu\} \otimes \{\lambda\} = C_{\mu\lambda}^{\nu} \{\nu\}$$

may be checked by calculating the dimension  $f^{\{\mu\} \otimes \{\lambda\}}$  of the plethysm  $\{\mu\} \otimes \{\lambda\}$ . Considering the interpretation of the operation of plethysm in symmetric group (Robinson, 1949, 1961) we have

$$f^{\{\mu\} \otimes \{\lambda\}} = \frac{(w_{\mu} w_{\lambda})!}{(w_{\mu}!)^{w_{\lambda}} w_{\lambda}!} (f^{\{\mu\}})^{w_{\lambda}} f^{\{\lambda\}} \tag{3.44}$$

The plethysm is then checked by noting that

$$f^{\{\mu\} \otimes \{\lambda\}} = C_{\mu\lambda}^{\nu} f^{\{\nu\}} \tag{3.45}$$

As the examples of (3.40) - (3.42) let us compute

$$(\frac{A+B}{2}) \otimes \{2\}$$

and

$$(\frac{A+B}{2}) \otimes \{1^2\}$$

Let

$$2(C \otimes \{2\}) + C^2 = A \otimes \{2\} + B \otimes \{2\} + AB$$

(left (2.42), right (3.40))

so we have

$$C \otimes \{2\} = \frac{1}{2} (A \otimes \{2\} + B \otimes \{2\} + AB - C^2)$$

Similarly we have

$$C \otimes \{1^2\} = \frac{1}{2} (A \otimes \{1^2\} + B \otimes \{1^2\} + AB - C^2)$$

If we let  $D=A-B$ , and use (3.41) and (3.42) we get

$$D \otimes \{2\} = \frac{1}{2} (A \otimes \{2\} + B \otimes \{1^2\} - AB - D^2)$$

$$D \otimes \{1^2\} = \frac{1}{2} (A \otimes \{1^2\} + B \otimes \{2\} - AB - D^2)$$

Certain special cases of S-function plethysms arise which are now enumerated.

$$\{\lambda\} \otimes \{0\} = \{0\} \tag{3.46}$$

$$\{0\} \otimes \{\lambda\} = \{0\} \quad \text{for } p_\lambda=1 \tag{3.47}$$

$$= 0 \quad \text{otherwise}$$

$$\{\lambda\} \otimes \{1\} = \{1\} \otimes \{\lambda\} = \{\lambda\} \tag{3.48}$$

If  $n$  is an integer, then

$$\{n\} \otimes \{2\} = \sum_k \{2(n-k), 2k\} \tag{3.49a}$$

$$\{n\} \otimes \{1^2\} = \sum_k \{2n-1-2k, 2k+1\} \quad (3.49b)$$

Use of the associated relation (3.43) then gives

$$\{1^n\} \otimes \{2\} = \sum_k \{2^{n-2k}, 1^{4k}\} \quad (3.50a)$$

$$\{1^n\} \otimes \{1^2\} = \sum_k \{2^{n-1-2k}, 1^{2+4k}\} \quad (3.50b)$$

Not all plethysms  $A \otimes \{\mu\}$  for different  $\{\mu\}$  are independent of one another. For example we have

$$A \otimes \{2\} = A^2 - A \otimes \{1^2\} \quad (3.51)$$

while

$$A \otimes \{3\} = (A \otimes \{2\})A - A \otimes \{2\} \quad (3.52a)$$

$$A \otimes \{1^3\} = (A \otimes \{1^2\})A - A \otimes \{21\} \quad (3.52b)$$

More significantly we have

$$(A \otimes \{2\}) \otimes \{2\} = A \otimes (\{2\} \otimes \{2\}) = A \otimes (\{4\} + \{2^2\}) \quad (3.53)$$

$$(A \otimes \{2\}) \otimes \{1^2\} = A \otimes (\{2\} \otimes \{1^2\}) = A \otimes \{31\} \quad (3.54)$$

$$(A \otimes \{1^2\}) \otimes \{2\} = A \otimes (\{1^2\} \otimes \{2\}) = A \otimes (\{2^2\} + \{1^4\}) \quad (3.55)$$

$$(A \otimes \{1^2\}) \otimes \{1^2\} = A \otimes (\{1^2\} \otimes \{1^2\}) = A \otimes \{21^2\} \quad (3.56)$$

and

$$A \otimes (\{3\}\{1\}) = (A \otimes \{3\})A = A \otimes (\{4\} + \{31\}) \quad (3.57)$$

$$A \otimes (\{1^3\}\{1\}) = (A \otimes \{1^3\})A = A \otimes (\{21^2\} + \{1^4\}) \quad (3.58)$$



$$A \otimes (\{2\}\{1\}) = (A \otimes \{2\})^2 = A \otimes (\{4\} + \{31\} + \{2^2\}) \quad (3.59)$$

leading to the basic identities

$$A \otimes \{4\} = (A \otimes \{3\})A - (A \otimes \{2\}) \otimes \{1^2\} \quad (3.60a)$$

$$A \otimes \{31\} = (A \otimes \{2\}) \otimes \{1^2\} \quad (3.60b)$$

$$A \otimes \{2^2\} = (A \otimes \{2\})^2 - (A \otimes \{3\})A \quad (3.60c)$$

$$A \otimes \{21^2\} = (A \otimes \{1^2\}) \otimes \{1^2\} \quad (3.60d)$$

$$A \otimes \{1^4\} = (A \otimes \{1^3\})A - (A \otimes \{1^2\}) \otimes \{1^2\} \quad (3.60e)$$

These allow all Kronecker powers of degree 4 to be resolved from a knowledge of lower degree powers. Indeed, to resolve all Kronecker powers of  $A$  of degree 2, 3 and 4, it is only necessary to evaluate  $A \otimes \{2\}$  and  $A \otimes \{21\}$  for example.

### 3.10 PLETHYSMS AND BRANCHING RULES

Wybourne (1970) gives the plethysm  $\{\mu\} \otimes \{\lambda\}$  a simple interpretation in terms of  $U_N \supset U_M$  ( $N > M$ ) group embedding. Let  $\{\mu\}$  be an S-function associated with an S-function associated with an irrep of  $U_M$  dimension  $D_M\{\mu\} = N$ , then an embedding of  $U_M$  in  $U_N$  may be defined by the mapping

$$\{1\} \downarrow \{\mu\} \quad (3.61)$$

Under this mapping an irrep  $\{\lambda\}$  of  $U_N$  will decompose into a sum of irreps of  $\{\nu\}$  of  $U_M$ . According to the definition of

the plethysm the branching rule is

$$\{\lambda\} \downarrow \{\mu\} \times \{\lambda\} = C_{\mu\lambda}^{\nu} \{\nu\} \quad (3.62)$$

The coefficients  $C_{\mu\lambda}^{\nu}$ , the branching multiplicities, will be independent of  $N$  and  $M$  provided  $p_{\nu} \leq M$ . A determination of these coefficients corresponds to the evaluation of the plethysm  $\{\mu\} \otimes \{\lambda\}$ .

As an example consider the canonical embedding of  $U_{M-1} \subset U_M$  defined by

$$\{1\} \downarrow \{1\} + \{0\} \quad (3.63)$$

Noting (3.62) we have the  $U_M \downarrow U_{M-1}$  branching rule

$$\{\lambda\} \downarrow (\{1\} + \{0\}) \otimes \{\lambda\} \quad (3.64)$$

Use of (3.40) followed by (3.47) and (3.48) yields the branching rule as

$$\{\lambda\} = \downarrow \sum \{\lambda/m\} = \{\lambda/M\} \quad (3.65)$$

By way of illustration

$$\begin{aligned} \{321\} \downarrow \{321/0\} + \{321/1\} + \{321/2\} + \{321/3\} \\ = \{321\} + \{32\} + \{31^2\} + \{31\} + \{2^21\} + \{2^2\} \\ + \{21^2\} + \{2\} \end{aligned}$$

In the case of  $U_3 \downarrow U_2$  all partition  $(\nu)$  with  $p_\nu > q$  must be removed leaving for  $U_3 \downarrow U_2$

$$\{321\} \downarrow \{32\} + \{31\} + \{2^2\} + \{21\} \quad .$$

In general for  $U_M \downarrow U_{M-1}$  the irrep  $\{\lambda\}$  of  $U_M$  decomposes into the sum of all irreps  $\{\lambda'\}$  of  $U_{M-1}$  that satisfy the restriction

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \dots \lambda'_{M-1} \geq \lambda_M$$

(3.62) can be generalized to any group  $G$  that possesses unitary reps (the case of compact and finite group). Let  $T$  be a unitary rep of  $G$  and  $D(T) = N$ . With respect to the rep  $T$   $G$  can be regarded as the subgroup of  $U_N$  under the embedding

$$\{1\} \downarrow T \tag{3.66}$$

If  $\{\lambda\}$  is an irrep of  $U_N$  then

$$\{\lambda\} \downarrow T \otimes \{\lambda\} = m_{T\lambda}^S S \tag{3.67}$$

gives the branching rule of  $U_N \downarrow G$ , where  $S$  are the irreps of  $G$ . If we denote the inverse of the embedding  $U_N \downarrow G$  by  $G \uparrow U_N$ , this result may be written in the following way

$$T \uparrow b_T^\rho \{\rho\} \tag{3.68}$$

The inverse may not exist and if it does the inverse may not be unique (King 1975). Suppose (3.68) exists and insert (3.68) into (3.67) and use the rule of the plethysm of S-function we can get the decomposition of (3.67).

### 3.11 INNER PLETHYSMS OF S-FUNCTIONS

The ordinary (or outer) plethysm of S-functions can be associated with the outer product of S-function such that

$$\{\mu\}^r = \sum f^{[\lambda]} \{\mu\} \otimes \{\lambda\}$$

A similar resolution of the inner product of S-functions may be such that

$$\begin{aligned} \{\mu\}^{\circ r} &= \{\mu\} \cdot \{\mu\} \circ \dots \circ \{\mu\} \quad (r \text{ times}) \\ &= \sum f^{[\lambda]} \{\mu\} \otimes \{\lambda\} . \end{aligned} \quad (3.69)$$

where  $\otimes$  denotes the operation of inner plethysm and  $w_\mu = n$ .  $\{\mu\} \otimes \{\lambda\}$  is the rep of  $S_n$  and generally reducible, so we have

$$\{\mu\} \otimes \{\lambda\} = d_{\mu\lambda}^v \{\nu\} \quad (3.70)$$

where  $\{\nu\}$  are partition of  $n$ . (3.70) can be interpreted as the branching rule of  $S_n$ . Let  $f^{[\mu]} = N$ , then  $[\mu] \subset U_N$ , the branching rule.

$$U_n \downarrow S_n$$

$$\{\lambda\} \downarrow \{\mu\} \odot \{\lambda\} \quad (3.71)$$

we have the relation

$$\{\mu\} \odot \{\lambda\} = [\mu] \otimes \{\lambda\} \quad (3.72)$$

between the inner plethysm and outer plethysm.

The inner plethysm ( $\odot$ ) follows an algebra similar, but not equivalent, to that of ordinary plethysm ( $\otimes$ ). Thus

$$\{\mu\} \odot (A \pm B) = \{\mu\} \odot A \pm \{\mu\} \odot B \quad (3.73)$$

$$\{\mu\} \odot (A \circ B) = (\{\mu\} \odot A) \circ (\{\mu\} \odot B) \quad (3.74)$$

$$(A \odot B) \odot C = A \odot (B \odot C) \quad (3.75)$$

$$(A + B) \odot \{\lambda\} = \sum A \odot \{\lambda/\mu\} \circ B \odot \{\mu\} \quad (3.76)$$

$$(A - B) \odot \{\lambda\} = - \sum (-1)^{w_\mu} A \odot \{\lambda/\mu\} \circ B \odot \{\tilde{\mu}\} \quad (3.77)$$

$$(A \circ B) \odot \{\lambda\} = \sum A \odot (\{\lambda\} \circ \{\mu\}) \circ (B \odot \{\mu\}) \quad (3.78)$$

$$\begin{aligned} \{\lambda\} \odot \{\mu\} &= \{\tilde{\lambda}\} \odot \{\mu\} & w_\lambda \text{ even} \\ &= \{\tilde{\lambda}\} \odot \{\tilde{\mu}\} & w_\lambda \text{ odd} \end{aligned} \quad (3.79)$$

Littlewood (1958a) has derived the special result

$$\{n-1, 1\} \odot \{1^r\} = \{n-1, r^r\} \quad (3.80)$$

which may be useful in our later discussion.

The dimension of inner plethysm  $\{\mu\} \odot \{\lambda\}$  is given (Robinson 1961)

$$f^{\{\mu\} \odot \{\lambda\}} = \frac{f^{[\lambda]}}{n!} G_N^{(\lambda)} \quad (3.81)$$

where  $N = f^{[\mu]}$  and  $G_N^{(\lambda)}$  is as given in (2.65) we have the useful dimension check

$$f^{\{\mu\} \odot \{\lambda\}} = d_{\mu\lambda}^{\nu} f^{[\nu]} \quad (3.82)$$

### 3.12 CLASSIFICATION OF IRREPS

Let  $T$  be a rep. of group  $G$ , then  $T^*$ , complex conjugate, is also a rep. of  $G$ . If  $T$  is irrep and unitary then so is  $T^*$ . However, the unitary irreps  $T$  and  $T^*$  are not necessarily equivalent. If they are equivalent then there is a unitary matrix  $U$  such that

$$T = UT^*U^{-1} \quad (3.83)$$

where  $U$  is either symmetric or antisymmetric. If  $U$  is symmetric, then a matrix  $D$  can be found such that the rep

$$T' = DTD^{-1} \quad (3.84)$$

is real. If  $U$  is antisymmetric then no matrix  $D$  has the above properties.

when  $T$  and  $T^*$  are equivalent and (8.83) holds,  $T$  is said to be real and orthogonal if  $U$  is symmetric and

real and symplectic if  $U$  is antisymmetric. In both cases the character is real but whereas for orthogonal rep the rep is equivalent to a real rep, in the case of symplectic rep the rep is not equivalent to a real rep.

Since

$$|U| = |U^t| \quad (3.85)$$

and

$$|-U| = (-1)^n |U| \quad (3.86)$$

where  $n$  is the dimension of the rep. we must conclude that symplectic rep. can only arise for even dimensional reps.

We have just seen that three distinct cases exist for reps  $T$  which may all be related to the existence of a matrix  $U$  such that

$$U = C_T U^t \quad (3.87)$$

where

$$\left. \begin{array}{l} C_T = 0 \text{ complex reps} \\ C_T = 1 \text{ orthogonal reps} \\ C_T = -1 \text{ symplectic reps} \end{array} \right\} \quad (3.88)$$

The quantity  $C_\lambda$  is related to the characters of  $G$  being the Frobenius-Schur invariant (Hamermesh 1962)

$$C_T = \frac{1}{g} \sum_{R \in G} \chi^T(R^2) \quad (3.89)$$

The Frobenius-Schur invariant  $C_T$  may be determined by an analysis of the Kronecker square of  $T$  to give

$$\left. \begin{aligned} C_T &= 1 && \text{if } T \otimes \{2\} \supset e \\ C_T &= -1 && \text{if } T \otimes \{1^2\} \supset e \\ C_T &= 0 && \text{if } T^2 \not\supset e \end{aligned} \right\} \quad (3.90)$$

where  $e$  is the identity irrep.

The Kronecker product of an orthogonal irrep with a symplectic irrep is necessarily symplectic. The product of two symplectic or two orthogonal irreps is orthogonal however the resulting orthogonal rep will be usually reducible and thus may contain pairs of symplectic irreps.

The classification of irreps also has important consequences in determining branching rules. Thus for example, if an orthogonal or symplectic irrep of a group  $G$  is restricted to a subgroup  $H$  then complex irreps of  $H$  must occur in complex conjugate pairs.

Let  $T$  be a rep of  $G$  and suppose  $G$  has an alternating rep  $\epsilon$ . Consider the symmetrized Kronecker square of  $T$ . Let  $S$  be its symmetric part and  $A$  its antisymmetric part so that

$$T^2 = S + A \quad (3.91)$$

then  $\epsilon S$  and  $\epsilon A$  are the reps of  $G$  and also symmetric and antisymmetric. Generally

$$T = S + A \neq \epsilon S + \epsilon A \quad (3.92)$$



so  $\epsilon S$  and  $\epsilon A$  are not the symmetric and antisymmetric part of Kronecker square. If  $T$  is self-associated and satisfies

$$T^2 = S + A = \epsilon S + \epsilon A \quad (3.93)$$

then  $\epsilon S$  and  $\epsilon A$  are also the symmetric and antisymmetric part. Thus in such a case, we have two choices

$$(1) \quad A \quad \text{and} \quad S \quad (3.94a)$$

$$(2) \quad \epsilon A \quad \text{and} \quad \epsilon S \quad (3.94b)$$

as the resolution of the Kronecker square. If in the first choice  $T$  is orthogonal (or symplectic) then in the second choice  $T$  is symplectic (orthogonal) i.e.  $T$  is either a orthogonal or symplectic dependent choice of (3.94). This may occur in projective rep. of  $G$  because it is a multivalued rep.

## CHAPTER 4

### THE REPRESENTATIONS OF THE ORTHOGONAL AND ROTATION GROUPS

#### 4.1 ORTHOGONAL GROUPS $O_N$ AND ROTATION GROUPS $SO_N$

The group of real transformations that leaves the quadratic form

$$\sum_{i=1}^N x_i^2 \tag{4.1}$$

invariant is known as the orthogonal group in  $N$  dimensions and is designated here as  $O_N$ . Its elements are the set of all real  $N \times N$  matrices  $A$  such that

$$AA^t = I_N \tag{4.2}$$

clearly it is a subgroup of  $U_N$ .

Since, by the definition of an orthogonal matrix  $AA^t = I_N$ , we have

$$|A| = \pm 1 \tag{4.3}$$

Thus the set of orthogonal matrices that characterize  $O_N$  may be divided into two subsets, those with determinant  $+1$  and those with determinant  $-1$ .

The subset of orthogonal matrices of determinant  $+1$  clearly form a subgroup of  $O_N$  which is known as the rotation group (or special orthogonal group) and is designated here as  $SO_N$ .

#### 4.2 THE REPRESENTATIONS OF $O_N$

When we regard  $O_N$  as the subgroup of  $U_N$  the reps in terms of tensors of a given symmetry  $\{\lambda\}$  will no longer be irreducible. The reason for this is that, in addition to the operation of symmetrization which we used for constructing irreps of  $U_N$ , a new operation of contraction appears which commutes with the orthogonal transformations. Using contraction we can show that every tensor  $F_{ij} \dots$  can be decomposed uniquely into a traceless tensor  $F^\circ$  plus a tensor  $\Phi$  (Hamermesh 1962) we can start from the subspace of traceless tensors and apply the symmetrization to obtain traceless tensors of a given symmetry type. In this way we arrive at the irreps of  $O_N$ . Thus each irrep  $\{\lambda\}$  of  $U_N$  **may** be a reducible rep of  $O_N$ . The irreps of  $O_N$  may also be labelled by partition  $(\lambda)$  and designated by enclosing the partition in square brackets, e.g.  $[\lambda]$ .

The relationships between the irreps of  $U_N$  and  $O_N$  are (Littlewood, 1950, Wybourne, 1970, King, 1975)

$$U_N \downarrow O_N \quad \{\lambda\} \downarrow [\lambda/D] \quad (4.4a)$$

$$O_N \uparrow U_N \quad [\lambda] \uparrow \{\lambda/C\} \quad (4.4b)$$

By way of example we have

$$\begin{aligned} \{321\} \downarrow [321/D] &= [321/0] + [321/2] + [321/2^2] \\ &= [321] + [31] + [2^2] + [21^2] + [2] + [1^2] \end{aligned}$$

$$\begin{aligned}
[321] + \{321/C\} &= \{321/0\} - \{321/2\} + \{321/31\} \\
&= \{321\} - \{31\} - \{2^2\} - \{21^2\} + \{2\} + \{1^2\}
\end{aligned}$$

The reps labelled by the partition of integers  $[\lambda]$  are often referred to as tensor irreps of  $O_N$ . In addition to tensor irreps there are the spin irreps. We denote the irreps of the spin irreps of  $O_N$  by  $[\Delta; \lambda]$  (King 1975b) where  $(\lambda)$  is a partition associated with a given symmetric type tensor and  $\Delta$  is a symbol associated with a spin index. The notation for the basic spin irrep will frequently be contracted to just  $\Delta = [\Delta; 0]$ . This irrep of  $O_N$  is of dimension  $2^\nu$  for both  $n = 2\nu$  and  $n = 2\nu+1$ .

$O_N$  has two one-dimensional irreps, the identity irrep  $[0]$  and the alternating irrep designated by  $[0]^\dagger$  whose character is +1 for transformations of positive determinant and -1 for transformations of negative determinant. If  $[\lambda]$  and  $[\Delta; \lambda]$  denotes any irreps of  $O_N$  then

$$[\lambda]^\dagger = [\lambda][0]^\dagger \quad (4.5a)$$

(or

$$[\Delta; \lambda]^\dagger = [\Delta; \lambda][0]^\dagger \quad (4.5b)$$

is also an irrep of  $O_N$  called the associated irrep of  $[\lambda]$  (or  $[\Delta; \lambda]$ ). It will differ from  $[\lambda]$  (or  $[\Delta; \lambda]$ ) unless the characters of all transformations of negative determinant are zero. If  $[\lambda]^\dagger = [\lambda]$  (or  $[\Delta; \lambda]^\dagger = [\Delta; \lambda]$ ) then  $[\lambda]$  (or  $[\Delta; \lambda]$ ) is said to be self-associated and will be designated as  $[\lambda]^\dagger$  (or  $[\Delta; \lambda]^\dagger$ ). Since for  $O_{2\nu}$   $[0; \lambda]$  is always self-associated we shall omit the  $\dagger$  for all spin irreps.

The following equivalent relation or modification rule holds for  $O_N$  (King, 1971, 1975b).

$$[\lambda] = (-1)^{x-1} [\lambda-h]^{\dagger} \quad \text{with } h = 2p_{\lambda} - n \quad (4.6a)$$

$$[\Delta; \lambda] = (-1)^x [\Delta; \lambda-h]^{\dagger} \quad \text{with } h = 2p_{\lambda} - n - 1 \quad (4.6b)$$

where  $h$  is the length of the continuous boundary hook removed from Young diagram corresponding to  $(\lambda)$  starting from the foot of the first column and ending in the  $x$ th column. The corresponding rep vanishes identically unless after the hook removal a regular Young diagram. These equivalences allow us to restrict ourselves to partitions of at most  $v$  parts for  $O_N$  both  $N = 2v$  and  $N = 2v+1$ , and to establish that the standard inequivalent irrep of  $O_N$  are those

$$O_{2v} \quad [\lambda], [\lambda]^{\dagger} \quad \text{for } p_{\lambda} < v \quad (4.79)$$

$$[\lambda]^{\dagger} \quad \text{for } p_{\lambda} = v \quad (4.7b)$$

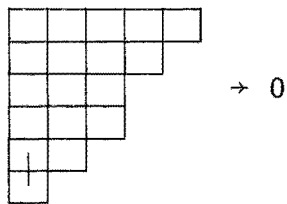
$$[\Delta; \lambda] \quad \text{for } p_{\lambda} \leq v \quad (4.7c)$$

and

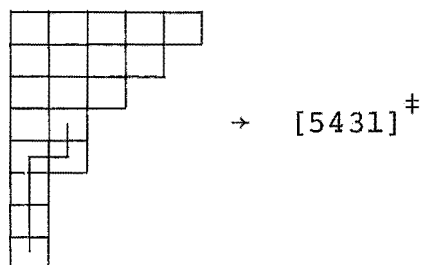
$$O_{2v+1} \quad [\lambda], [\lambda]^{\dagger} \quad \text{for } p_{\lambda} \leq v \quad (4.8a)$$

$$[\Delta; \lambda], [\Delta; \lambda]^{\dagger} \quad \text{for } p_{\lambda} \leq v \quad (4.8b)$$

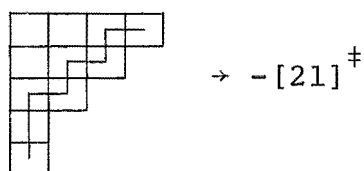
Consider the  $[543^2 21]$  of  $O_{10}$ , then  $h = 2p - n = 12 - 10 = 2$




the resulting Young diagram  $[\lambda-h]$  is irregular, so the  $[543^2 21]$  of  $O_{10}$  vanishes while for  $[5432^2 1^3]$  of  $O_{10}$  we have  $h = 16 - 10 = 6$  and  $x = 2$  leading to



When we use the modification rule to  $[4321^2]$  of  $O_2$  and have  $h = 8$ ,  $x = 4$  hence



but  $[21]^\dagger$  is a non-standard irrep for  $O_2$ . However for  $[21]$  of  $O_2$  we have  $h = 1$  and  $x = 1$  leading to   $\rightarrow [2]^\dagger$  and hence for  $O_2$ ,  $[4321^2] = -[2]$ . This last case shows that application of (4.6) will not always lead directly to a standard irrep but repeated application of (4.6) will eventually produce either a null result or a standard symbol.

As an example, we consider the branching rule  $U_4 \downarrow O_4$

$$\begin{aligned} \{321\} \downarrow & [321] + [31] + [2^2] + [21^2] + [2] + [1^2] \\ & = [31] + [2^2] + 2[2] + [1^2] \end{aligned}$$

because  $[321] = 0$  and  $[21^2] = [2]$  for  $O_4$  using the modification rule.

The spin irreps  $[\Delta; \lambda]$  of  $O_N$  may be written as a product of the basic spin irrep  $\Delta$  and tensor irreps by noting (King 1975b) that for  $O_{2v}$

$$[\Delta; \lambda] = \Delta[\lambda/P] \quad (4.9a)$$

and inversely

$$\Delta[\lambda] = [\Delta; \lambda/Q] \quad (4.9b)$$

whilst for  $O_{2v+1}$

$$[\Delta; \lambda] = \Delta[\lambda/P^\dagger] \quad (4.10a)$$

and

$$\Delta[\lambda] = [\Delta; \lambda/Q^\dagger] \quad (4.10b)$$

where it has been convenient to define

$$P^\dagger = \sum_m (-1)^m \{m\}^{(\dagger)m} \quad (4.11a)$$

and

$$Q^\dagger = \sum_m \{1^m\}^{(\dagger)m} \quad (4.11b)$$

and it is to be understood that, for example

$$[\lambda/m^{(\dagger)m}] = [\lambda/m]^{(\dagger)m} = \begin{cases} [\lambda/m] & \text{if } m \text{ is even} \\ [\lambda/m]^\dagger & \text{if } m \text{ is odd} \end{cases} \quad (4.12a)$$

since of course

$$([0]^{(\dagger)})^m = [0]^{(\dagger)m} = \begin{cases} [0] & \text{if } m \text{ is even} \\ [0]^\dagger & \text{if } m \text{ is odd} \end{cases} \quad (4.12b)$$

Provided that the modification rules are used to ensure that these formulas are used only in the case for which  $(\lambda)$  is a partition consisting of  $q$  non-vanishing parts with  $q \leq v$ .

Consider  $[\Delta; 321]$  of  $O_{10}$ , then

$$\begin{aligned} [\Delta; 321] &= \Delta[321/P] \\ &= \Delta([321] - [2^2 1] - [31^2] - [34] + [21^2] \\ &\quad + [2^2] + [31] - [21]) \end{aligned}$$

while for  $O_{11}$  we have

$$\begin{aligned} [\Delta; 321] &= \Delta([321] - [2^2 1]^\dagger - [31^2]^\dagger - [32]^\dagger + [21^2] \\ &\quad + [2^2] + [31] - [21]^\dagger) \end{aligned}$$

#### 4.3 DIMENSIONS OF IRREPS OF $O_N$

Weyl (1926) has shown that the dimensions of  $O_N$  may be computed as

$$D_{2v}[\lambda] = \prod_{1 \leq i \leq j \leq v} \frac{(\lambda_i - \lambda_j - i + j)(\lambda_i + \lambda_j - i - j + 2v)}{(-i + j)(-i - j + 2v)} \quad (4.13a)$$

and

$$D_{2v+1}[\lambda] = \prod_{1 \leq i \leq j \leq v} \frac{(\lambda_i - \lambda_j - i + j)}{(-i + j)} \prod_{1 \leq i \leq j \leq v} \frac{(\lambda_i + \lambda_j - i - j + 2v + 1)}{(-i - j + 2v + 1)} \quad (4.13b)$$



Equations (4.13a) and (4.13b) hold for both tensor and spin irreps of  $O_N$ . In the case of spin irreps  $[\Delta; \lambda]$  the  $\lambda_k$ 's appearing in (4.13) must be replaced by  $\lambda_k + \frac{1}{2}$ . For  $O_N$  with  $N = 2\nu$  the results given in (4.13a) must be doubled for the tensor irreps of  $\lambda_\nu \neq 0$  and for all spin irreps.

The above formulas are somewhat cumbersome and give no clue as to the dependence of the dimension of an irrep  $[\lambda]$  or  $[\Delta; \lambda]$  upon  $N$ . El Samra and King (1979) have produced formulas that overcome this deficiency. For the tensor irreps of  $O_N$  they find

$$D_N[\lambda] = \prod_{i \geq j}^{\lambda} (N + \lambda_i + \lambda_j - i - j) \prod_{i < j}^{\lambda} (N - \tilde{\lambda}_i - \tilde{\lambda}_j + i + j - 2) / H(\lambda) \quad (4.14)$$

and for the spin irreps

$$D_N[\Delta; \lambda] = 2^\nu \prod_{i > j}^{\lambda} (N + \lambda_i + \lambda_j - i - j + 1) \prod_{i \leq j}^{\lambda} (N - \tilde{\lambda}_i - \tilde{\lambda}_j + i + j - 1) / H(\lambda) \quad (4.15)$$

where the products are taken over all pairs  $(i, j)$  specifying positions of cells of Young diagram corresponding to the partition  $(\lambda)$ .

The above results display explicitly the  $N$ -dependence of the dimensional formulas and can be used to write down general results. Thus if  $[\lambda] = [31]$  we have from (4.14) the general result.

$$D_N[31] = \frac{1}{8} (N+4) (N+1) (N-2) (N-1)$$

and hence

$$D_4[31] = 30, D_5[31] = 81, D_{10}[31] = 1396 \text{ etc}$$

Likewise for  $[\Delta; 31]$  we have from (4.15)

$$D_N[\Delta; 31] = \frac{2^v}{8} (N+2)N(N-1)(N-3)$$

and hence

$$D_4[\Delta; 31] = 36, D_5[\Delta; 31] = 140, D_{10}[\Delta; 31] = 30240$$

A short list of dimensions of  $O_{10}$  irreps is given in Table III.

#### 4.4 BRANCHING RULES FOR $O_N \downarrow O_{N-1}$

There is no difficulty in determining branching rules for  $O_N \downarrow O_{N-1}$  once the reductions of the vector irrep  $[1]$  is fixed. Thus noting that under  $O_N \downarrow O_{N-1}$

$$[1] \downarrow [1] + [0]$$

we readily deduce that

$$[\lambda] \downarrow ([1] + [0]) \otimes [\lambda]$$

But

$$\begin{aligned} ([1] + [0]) \otimes [\lambda] &= (\{1\} + \{0\}) \otimes \{\lambda/C\} \\ &= \{\lambda/CM\} = [\lambda/CMD] = [\lambda/M] \end{aligned}$$

Since  $CD = 1$ .

TABLE III: Dimensions of tensor and spin irreps of  $O_{10}$ 

$[\lambda]$	$D_{10}[\lambda]$	$D_{10}[\Delta; \lambda]$
$[0]$	1	32
$[1]$	10	288
$[1^2]$	45	1120
$[1^3]$	120	2400
$[1^4]$	210	2880
$[1^5]$	252	1344
$[2]$	54	1420
$[21]$	320	7392
$[21^2]$	945	17600
$[21^3]$	1728	22176
$[21^4]$	2100	10560
$[2^2]$	770	16128
$[2^21]$	1970	50400
$[2^21^2]$	5940	69984
$[2^21^3]$	7392	34560
$[2^3]$	4125	61600
$[2^31]$	10560	110880
$[2^31^2]$	13860	59136
$[2^3]$	8910	79200
$[2^31]$	13860	52800
$[2^5]$	5544	19008

Thus under  $O_N \downarrow O_{N-1}$

$$[\lambda] \downarrow [\lambda/M] \quad (4.16)$$

So far we have restricted our development of branching rules to the tensor irreps of  $O_N$ . The branching rules for spin irreps  $[\Delta; \lambda]$  of  $O_N$  follow easily by first noting that  $[\Delta; \lambda]$  may be expanded as a sum of Kronecker products of the basic spin irreps  $\Delta$  with tensor irreps  $[p]$  by noting (4.9) and (4.10).

If the branching rules for the basic spin irrep  $\Delta$  is known then there is little difficulty in obtaining the branching rules for spin irreps of the form  $[\Delta; \lambda]$ .

$$O_{2N} \downarrow O_{2N-1} \quad \Delta \downarrow \Delta + \Delta^\ddagger \quad (4.17a)$$

and

$$O_{2N+1} \downarrow O_{2N} \quad \begin{matrix} \Delta \\ \Delta^\ddagger \end{matrix} \downarrow \Delta \quad (4.17b)$$

Noting the above results readily leads to the branching rules

$$O_{2N} \downarrow O_{2N-1} \quad [\Delta; \lambda] \downarrow [\Delta; \lambda/M] + [\Delta; \lambda/M]^\ddagger \quad (4.18a)$$

and

$$O_{2N+1} \downarrow O_{2N} \quad \begin{matrix} [\Delta; \lambda] \\ [\Delta; \lambda]^\ddagger \end{matrix} \downarrow [\Delta; \lambda/M] \quad (4.18b)$$

It is necessary to use modification rules (4.6) if non-standard symbols occur in the above expansions.

As the example of (4.16) we consider the branching rule of  $[321]$  of  $O_N$  to  $O_{N-1}$ .

$$\begin{aligned} [321] \downarrow [321/M] &= [321] + [2^2 1] + [31] + [32] \\ &+ [21^2] + [2^2] + [31] + [21] \end{aligned}$$

If  $N = 8$  all the symbols are standard so we have  $O_8 \downarrow O_7$

$$\begin{aligned} [321] \downarrow [321] &+ [2^2 1] + [31^2] + [32] + [21^2] + [2^2] \\ &+ [31] + [21] \end{aligned}$$

If  $N = 6$  then  $O_6 \downarrow O_5$

$$\begin{aligned} [321] \downarrow [32]^\dagger &+ [2^2]^\dagger + [31]^\dagger + [32] + [21]^\dagger + [2^2] \\ &+ [31] + [21] \end{aligned}$$

while the branching rule of  $[\Delta; 321]$  of  $O_{2v+1}$  to  $O_{2v}$  is

$$\begin{aligned} [\Delta; 321] \downarrow [\Delta; 321/M] &= [\Delta; 321] + [\Delta; 2^2 1] + [\Delta; 31^2] + [\Delta; 32] \\ &+ [\Delta; 21^2] + [\Delta; 2^2] + [\Delta; 31] + [\Delta; 21] \end{aligned}$$

while the branching rule of  $[\Delta; 321]$  of  $O_{2v}$  to  $O_{2v-1}$  is

$$[\Delta; 321] \downarrow [\Delta; 321] + [\Delta; 2^2 1] + [\Delta; 31^2]$$

$$\begin{aligned}
& + [\Delta; 32] + [\Delta; 21^2] + [\Delta; 2^2] \\
& + [\Delta; 31] + [\Delta; 21] + [\Delta; 321]^{\dagger} \\
& + [\Delta; 2 \ 1]^{\dagger} + [\Delta; 31 \ ]^{\dagger} + [\Delta; 32 \ ]^{\dagger} \\
& + [\Delta; 21^2]^{\dagger} + [\Delta; 2^2] + [\Delta; 31]^{\dagger} + [\Delta; 21]^{\dagger}
\end{aligned}$$

and application to  $O_6 \downarrow O_5$  then gives

$$\begin{aligned}
& [\Delta; 321] \downarrow [\Delta; 32] + [\Delta; 2^2] + [\Delta; 31] + [\Delta; 21] \\
& + [\Delta; 32]^{\dagger} + [\Delta; 2^2]^{\dagger} + [\Delta; 31]^{\dagger} + [\Delta; 21]^{\dagger}
\end{aligned}$$

#### 4.5 KRONECKER PRODUCTS OF THE IRREPS OF $O_N$

##### I. Kronecker products of Tensor irreps

The Kronecker products for the tensor irreps of  $O_N$  may be evaluated by use of (4.4a) to expand the irreps as S-function products formed and then the S-functions expanded back into irreps of  $O_N$  (using (4.4b)) as appropriate to give the results for  $O_N$  (King 1975)

$$[\lambda][\mu] = [((\lambda/C) \cdot (\mu/C))/D] \quad (4.19)$$

where  $(\lambda/C) \cdot (\mu/C)$  represents the outer product of S-function  $\{\lambda/C \cdot \mu/C\}$ . However this rule is exactly equivalent to the much simpler rule also given by Littlewood (1958)

$$O_N [\lambda][\mu] = \sum_{\zeta} [(\lambda/\zeta) \cdot (\mu/\zeta)] \quad (4.20)$$

For the special case  $[1^n][1^n]$  we have

$$\begin{aligned}
 [1^n][1^{n'}] &= \sum_{\zeta} [(1^n/\zeta) \cdot (1^{n'}/\zeta)] \\
 &= \sum_{q=0}^{\min(n,n')} [(1^{n-q}) \cdot (1^{n'-q})] \quad (4.21)
 \end{aligned}$$

By way of example we have for the product

$$\begin{aligned}
 [21][1^2] &= \sum_{\zeta} [(21/\zeta) \cdot (1^2/\zeta)] \\
 &= [(21) \cdot (1^2)] + [(2) \cdot (1)] + [(1^2) \cdot (1)] + [(1) \cdot (0)] \\
 &= [32] + [31^2] + [3] + [2^21] + [21^3] + 2[21] \\
 &\quad + [1^3] + [1]
 \end{aligned}$$

Where the number of variables is restricted. Any non-standard symbols must be converted into standard form using the modification rule (4.6a).

So the above example for  $O_5$  is

$$\begin{aligned}
 [21] \cdot [1^2] &= [32] + [31]^{\dagger} + [3] + [22]^{\dagger} + [2]^{\dagger} + 2[21] \\
 &\quad + [1^2]^{\dagger} + [1]
 \end{aligned}$$

## II. Kronecker Products Of Spin Irreps

The resolution of the square of the basic spin irreps  $\Delta$  of  $O_N$  are well-known from the work of Brauer and Weyl (1935) and may be conveniently written in the form:

$$O_{2v} \Delta^2 = \sum_{r=0}^v [1^r] = [1^v] + \sum_s [1^{v-1-s}] + [1^{v-1-s}]^\dagger \quad (4.22a)$$

$$O_{2v+1} \Delta^2 = \sum_{r=0}^v [1^r] = \sum_s [1^{v-s}]^{(\dagger) v-s} \quad (4.22b)$$

where use has been made of the equivalence  $[1^r] = [1^{n-r}]^\dagger$  which follows from (4.6a).

(4.22) may also be written as

$$O_{2v} \Delta^2 = Q = Q^\dagger \quad (4.23a)$$

$$O_{2v+1} \Delta^2 = \frac{1}{2} Q^\dagger \quad (4.23b)$$

Since  $[1^r] = [1^{N-r}]^\dagger = \{1^r\}$  and  $\{1^r\} = 0$  for  $r > N$ .

It is then straightforward to show that for  $O_{2v}$

$$[\Delta; \lambda] [\Delta; \mu] = \sum_{\zeta} [Q \cdot (\lambda/\zeta) \cdot (\mu/\zeta)] \quad (4.24a)$$

and for  $O_{2v+1}$

$$[\Delta; \lambda] [\Delta; \mu] = \sum_{\zeta} \frac{1}{2} [Q^\dagger \cdot (\lambda/\zeta) \cdot (\mu/\zeta)] \quad (4.24b)$$

The modification rules limit the number of terms that appear in (4.24a) and (4.24b). The modification rules appropriate to any term  $(\rho)$  appearing in the products  $(\lambda/\zeta) \cdot (\mu/\zeta)$  of (4.24a) and (4.24b) are

$$(\rho) = (-1)^{x-1} (\rho-h)^\dagger \quad \text{with} \quad h = 2p_\rho - 2v - 1 \quad (4.25a)$$

and

$$(\rho) = (-1)^{x-1} (\rho-h) \quad \text{with} \quad h = 2p_\rho - 2v - 2 \quad (4.25b)$$



respectively; making use of these rules to restrict consideration to cases for which  $p_\rho \leq v$  for  $O_{2v}$

$$[Q \cdot \rho] = [1^v; \rho/Q] + \sum_{s=0}^{v-1-p_\rho} ([1^{v-1-s}; \rho/Q] + [1^{v-1-s}; \rho/Q]^\dagger) \quad (4.26a)$$

and for  $O_{2v+1}$

$$\frac{1}{2}[Q^\dagger \cdot \rho] = \sum_{s=0}^{v-p_\rho} [1^{v-s}; \rho/Q^\dagger]^{(\dagger)(v-s)} \quad (4.26b)$$

It then follows that (4.24a) and (4.24b) may be replaced for  $O_{2v}$  by

$$\begin{aligned} [\Delta; \lambda][\Delta; \mu] &= \sum_{\zeta} \{ [1^v; ((\lambda/\zeta) \cdot (\mu/\zeta))_{p_\rho}/Q] \\ &+ \sum_{s=0}^{v-1-p_\rho} ([1^{v-1-s}; (\lambda/\zeta) \cdot (\mu/\zeta))_{p_\rho}/Q] \\ &+ [1^{v-1-s}; ((\lambda/\zeta) \cdot (\mu/\zeta))_{p_\rho}/Q]^\dagger \} \end{aligned} \quad (4.27a)$$

and for  $O_{2v+1}$ , by

$$[\Delta; \lambda][\Delta; \mu] = \sum_{\zeta} \{ \sum_{\zeta=0}^{v-p_\rho} [1^{v-s}; (\lambda/\zeta) \cdot (\mu/\zeta))_{p_\rho}/Q^\dagger]^{(\dagger)(v-s)} \} \quad (4.27b)$$

where the subscript  $p_\rho$  has been included in the factor  $((\lambda/\zeta) \cdot (\mu/\zeta))_{p_\rho}$  as a reminder that each term  $(\rho)$  in this factor must be modified, using (4.25a) and (4.25b), to give a term specified by a partition into  $p_\rho$  parts with  $p_\rho \leq v$ . This has to be carried out prior to the division by  $Q$  or

$Q^\dagger$ . No further modification rules are then necessary.

### III. Kronecker Products Of Tensor And Spin Irreps

It is now a comparatively simple matter to deduce that  
for  $O_{2v}$

$$[\Delta; \lambda][\mu] = \sum_{\zeta} [\Delta; (\lambda/\zeta) \cdot (\mu/\zeta Q)] \quad (4.28a)$$

and for  $O_{2v+1}$

$$[\Delta; \lambda][\mu] = \sum_{\zeta} [\Delta; (\lambda/\zeta) \cdot (\mu/\zeta Q^\dagger)] \quad (4.28b)$$

#### 4.6 PLETHYSM OF $O_N$

The key to the plethysm of irreps of  $O_N$  is the relationships between  $O_N$  and  $U_N$  (4.4a) and (4.4b). So for tensor irreps of  $O_N$  we have

$$[\lambda] \otimes \{\mu\} = [(\{\lambda/C\} \otimes \{\mu\})/D] \quad (4.29a)$$

and

$$[\lambda]^\dagger \otimes \{\mu\} = [(\{\lambda/C\} \otimes \{\mu\})/D]^{(\dagger)w_\mu} \quad (4.29b)$$

This last result follows from the facts that  $[\lambda]^\dagger = [0]^\dagger[\lambda]$ ,  $[0]^\dagger = [1^n] = \{1^n\}$  and  $\{1^n\} \otimes \{\rho\} = 0$  for  $(\rho) \neq (n)$  and  $\{1^n\} \otimes \{m\} = \{1^{nm}\} = \{1^n\}^m = [0]^\dagger{}^m$ , together with (3.51).

In the spin irreps of  $O_N$ , the plethysm may be got by recalling (3.51) and making use of (4.9a) and (4.10a) which lead for  $O_{2v}$  to

$$[\Delta; \lambda] \otimes \{\mu\} = \sum_{\zeta} [\Delta \otimes \{\zeta\}] [\{\lambda/E\} \otimes \{\mu \circ \zeta\}/D] \quad (4.30a)$$

and for  $O_{2v+1}$

$$[\Delta; \lambda] \otimes \{\mu\} = \sum_{\zeta} [\Delta \otimes \{\zeta\}] [\{\lambda/E^{\dagger}\} \otimes \{\mu \circ \zeta\}/D] \quad (4.30b)$$

where  $E^{\dagger} = D^{\dagger}C$ .

#### 4.7 SYMMETRIZED KRONECKER SQUARE OF IRREPS OF $O_N$

For the tensors irreps of  $O_N$  equations (4.29a) and (4.29b) suffer from severe overcounting but in the symmetrized Kronecker square case i.e.  $(\mu) = (2)$  or  $(1^2)$ , they may be simplified to give

$$[\lambda]^{\dagger} \otimes \{\mu\} = [\lambda] \otimes \{\mu\} = \sum_{\zeta} [\{\lambda/\zeta\} \otimes \{\mu\}]$$

$$\mu = (2) \text{ or } (1^2) \quad (4.31)$$

because  $\{0\} \otimes \{2\} = [0]$  (cf. 3.47) is always included in  $[\lambda] \otimes \{2\}$ , all tensor irreps  $[\lambda]$  of  $O_N$  are orthogonal.

As an example we have

$$\begin{aligned} [2] \otimes \{2\} &= [\{2\} \otimes \{2\}] + [\{1\} \otimes \{2\}] + [\{0\} \otimes \{2\}] \\ &= [4] + [2^2] + [2] + [0] \end{aligned}$$

and

$$\begin{aligned} [2] \otimes \{1^2\} &= [\{2\} \otimes \{1^2\}] + [\{1\} \otimes \{1^2\}] \\ &= [31] + [1^2] \end{aligned}$$

For the spin irreps of  $O_N$  from (4.30a) and (4.30b) we see that we need to know the plethysms of basic spin irreps.

The resolution of Kronecker square of the basic spin irreps  $\Delta$  of  $O_N$  is known from Littlewood (1947). For  $O_{2v}$ .

$$\begin{aligned} \Delta \otimes \{2\} = [1^v] + \sum_x ([1^{v-1-4x}] + [1^{v-3-4x}]^\dagger + [1^{v-4-4x}] \\ + [1^{v-4-4x}]^\dagger) \end{aligned} \quad (4.32a)$$

$$\begin{aligned} \Delta \otimes \{1^2\} = \sum_x ([1^{v-1-4x}]^\dagger + [1^{v-2-4x}] + [1^{v-2-4x}]^\dagger \\ + [1^{v-3-4x}]^\dagger) \end{aligned} \quad (4.32b)$$

whilst for  $O_{2v+1}$  the cases for  $v$  even and  $v$  odd must be distinguished, giving for  $v$  even

$$\Delta \otimes \{2\} = [1^v] + \sum_x ([1^{v-3-4x}]^\dagger + [1^{v-4-4x}]) \quad (4.33a)$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{v-1-4x}]^\dagger + [1^{v-2-4x}]) \quad (4.33b)$$

and for  $v$  odd

$$\Delta \otimes \{2\} = [1^v]^\dagger + \sum_x ([1^{v-3-4x}] + [1^{v-4-4x}]^\dagger) \quad (4.34a)$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{v-1-4x}] + [1^{v-2-4x}]^\dagger) \quad (4.34b)$$

But for  $O_{2v}$ ,  $\Delta$  is self-associated and satisfies the condition (3.10a) pointed out in chapter 3.

$$\Delta^2 = S+A = \epsilon S + \epsilon A$$

so we can have the second choice of resolution of the Kronecker square.

$$\begin{aligned} \Delta \otimes \{2\} = [1^v] + \sum_x ([1^{v-1-4x}] + [1^{v-3-4x}] + [1^{v-4-4x}] \\ + [1^{v-4-4x}]^\dagger) \end{aligned} \quad (4.35a)$$

$$\begin{aligned} \Delta \otimes \{1^2\} = \sum_x ([1^{v-1-4x}] + [1^{v-2-4x}] + [1^{v-2-4x}]^\dagger \\ + [1^{v-3-4x}]^\dagger) \end{aligned} \quad (4.35b)$$

In this thesis we take the second choice.

From (4.35a), (4.35b) and (4.33a) - (4.34b) we find for the basic spin  $\Delta$ ,

$$N = 0,1,6,7 \pmod{8} \quad \text{all orthogonal}$$

$$N = 2,3,4,5 \pmod{8} \quad \text{all symmetric}$$

We can always reduce any spin irrep to the product of the basic spin irrep with a sum of tensor irreps. The tensor irreps of  $O_N$  are orthogonal and hence once the classifications of the basic spin irrep of  $O_N$  is known, we can obtain the classification of any spin irrep of  $O_N$ .

So for  $O_N$  the tensor irreps  $[\lambda]$  are all orthogonal while for the spin irreps  $[\Delta; \lambda]$  we have

$N = 0, 1, 6, 7 \pmod{8}$	all orthogonal
$N = 2, 3, 4, 5 \pmod{8}$	all symplectic

#### 4.8 REPRESENTATIONS OF $SO_N$

Under  $O_N \downarrow SO_N$  the pair of associated irreps of  $O_N$  become equivalent irreps of  $SO_N$ , while self-associated irreps of  $O_N$  split into two conjugate irreps of  $SO_N$  of the same dimension. Thus we shall label the tensor irreps by  $[\lambda]$  and spin irreps by  $[\Delta; \lambda]$  for  $SO_{2v+1}$  while the tensor irrep by  $[\lambda]$  ( $p_\lambda < v$ ),  $[\lambda]_+$  and  $[\lambda]_-$  ( $p_\lambda = v$ ) and the spin irreps by  $[\Delta; \lambda]_+$  and  $[\Delta; \lambda]_-$  for  $SO_{2v}$ .

We then have the branching rules

$$O_{2v+1} \downarrow SO_{2v+1}$$

$$[\lambda] \downarrow [\lambda] \quad (4.36a)$$

$$[\lambda]^\pm \downarrow [\lambda] \quad (4.36b)$$

$$[\Delta; \lambda]^\pm \downarrow [\Delta; \lambda] \quad (4.36c)$$

and

$$O_{2v} \downarrow SO_{2v}$$

$$[\lambda] \downarrow [\lambda] \quad p_\lambda < v \quad (4.37a)$$

$$[\lambda]^\pm \downarrow [\lambda] \quad p_\lambda < v \quad (4.37b)$$

$$[\lambda]^\pm \downarrow [\lambda]_+ + [\lambda]_- \quad p_\lambda = v \quad (4.37c)$$

$$[\Delta; \lambda] \downarrow [\Delta; \lambda]_+ + [\Delta; \lambda]_- \quad p_\lambda \leq v \quad (4.37d)$$

The irreps  $[\lambda]_+$ ,  $[\lambda]_-$  and  $[\Delta; \lambda]_+$ ,  $[\Delta; \lambda]_-$  are conjugate to one another under an involutory outer automorphism\* of  $SO_{2\nu}$  involving a matrix of determinant -1. Under this automorphism

$$*[\lambda] = [\lambda] \quad p < \nu \quad (4.38a)$$

$$*[\lambda]_{\pm} = [\lambda]_{\mp} \quad p = \nu \quad (4.38b)$$

$$*[\Delta; \lambda]_{\pm} = [\Delta; \lambda]_{\mp} \quad p \leq \nu \quad (4.38c)$$

#### 4.9 DIFFERENCE IRREPS FOR $SO_{2\nu}$

In the case of inequivalent, but conjugate, pairs of  $SO_{2\nu}$  irreps it is convenient to denote the reps of their sums by

$$[\lambda]^+ = [\lambda]_+ + [\lambda]_- \quad (4.39a)$$

$$[\Delta; \lambda] = [\Delta; \lambda]_+ + [\Delta; \lambda]_- \quad (4.39b)$$

and the reps of their differences (Murnaghan 1938, Littlewood 1950, Wybourne 1970) by

$$[\lambda]^- = [\lambda]_+ - [\lambda]_- \quad (4.40a)$$

$$[\Delta; \lambda]^- = [\Delta; \lambda]_+ - [\Delta; \lambda]_- \quad (4.40b)$$

so that

$$[\lambda]_{\pm} = \frac{1}{2}([\lambda]^{\dagger} \pm [\lambda]^{*}) \quad (4.41a)$$

$$[\Delta; \lambda]_{\pm} = \frac{1}{2}([\Delta; \lambda] \pm [\Delta; \lambda]^{*}) \quad (4.41b)$$

We then have for  $SO_{2v}$  (Littlewood 1950)

$$[\Delta; \lambda] = \Delta[\lambda/p] \quad (4.42a)$$

$$[\Delta; \lambda]^{*} = \Delta^{*}[\lambda/M] \quad (4.42b)$$

and inversely

$$\Delta[\lambda] = [\Delta; \lambda/Q] \quad (4.43a)$$

$$\Delta^{*}[\lambda] = [\Delta; \lambda/L]^{*} \quad (4.43b)$$

Noting (4.41b) then leads to

$$[\Delta; \lambda]_{\pm} = \sum_m (-1)^m \Delta_{\pm}(-m)^m [\lambda/m] \quad (4.44a)$$

$$\Delta_{\pm}[\lambda] = \sum_m [\Delta; \lambda/1^m]_{\pm} (-1)^m \quad (4.44b)$$

In the case of tensor irreps of  $SO_{2v}$  it is convenient to write (El Samra and King 1979a)

$$\square = [\square; 0] = [1^v] = [1^v]_{+} + [1^v]_{-} \quad (4.46a)$$

$$\square^{*} = [\square; 0]^{*} = [1^v]_{+} - [1^v]_{-} \quad (4.46b)$$

$$\square_{\pm} = [\square; 0]_{\pm} = [1^v]_{\pm} = \frac{1}{2}(\square \pm \square^{*}) \quad (4.46c)$$



The analogues of (4.42) and (4.43) then become

$$[\square; \lambda] = \square[\lambda/Y] + 2 \sum_{s,t} (-1)^{t-1} [1^{v-1-t}] [\lambda/(1+t+s)/s] \quad (4.47a)$$

$$[\square; \lambda]'' = \square''[\lambda/W] \quad (4.47b)$$

and inversely

$$\square[\lambda] = [\square; \lambda/X] + 2 \sum_{s,t} [1^{v-1-t}; \lambda/1^{1+t+s}/1^s] \quad (4.48a)$$

$$\square''[\lambda] = [\square; \lambda/V]'' \quad (4.48b)$$

so that

$$[\square; \lambda]_{\pm} = \sum_{\omega} \square_{\pm}(-1)^q [\lambda/\omega] + \sum_{s,t} (-1)^{t-1} [1^{v-1-t}] [\lambda/(1+t+s)/s] \quad (4.49a)$$

$$\square_{\pm}[\lambda] = \sum_{\omega} [\square; \lambda/\tilde{\omega}]_{\pm}(-1)^q + \sum_{s,t} [1^{v-1-t}; \lambda/1^{1+t+s}/1^s] \quad (4.49b)$$

where use has been made of the notation introduced in sec.

2.9.

These results may all be derived from the work of Littlewood (1950) through the use of S-functions, and the rules for evaluating Kronecker products of tensor irreps of  $O_N$  (Littlewood 1958).

To conclude this section, it is worth pointing out that for irreps of  $SO_{2v}$  the complete set of modification rules is

$$[\lambda] = (-1)^{x-1} [\lambda-h] \quad \text{with } h = 2p_{\lambda} - 2v \quad (4.50a)$$

$$\left. \begin{aligned} [\Delta; \lambda] &= (-1)^X [\Delta; \lambda-h] \\ [\Delta; \lambda]'' &= (-1)^{X-1} [\Delta; \lambda-h]'' \\ [\Delta; \lambda]_{\pm} &= (-1)^X [\Delta; \lambda-h]_{\mp} \end{aligned} \right\} \quad \text{with } h = 2p_{\lambda} - 2v - 1 \quad (4.50b)$$

$$\left. \begin{aligned} [\square; \lambda] &= (-1)^{X-1} [\square; \lambda-h] \\ [\square; \lambda]'' &= (-1)^X [\square; \lambda-h]'' \\ [\square; \lambda]_{\pm} &= (-1)^{X-1} [\square; \lambda-h]_{\mp} \end{aligned} \right\} \quad \text{with } h = 2p_{\lambda} - 2v - 2 \quad (4.50c)$$

#### 4.10 KRONECKER PRODUCTS OF IRREPS OF $SO_{2v}$

For  $SO_{2v+1}$  we simply delete the  $\mp$  sign in the formulas of Kronecker products and modification rules for  $O_{2v+1}$ .

Hence here we need only consider the case of  $SO_{2v}$

##### I. Basic Kronecker products

First we consider the case of  $\square$ .

Putting  $[\lambda] = \square$  in (4.48) yields

$$\square^2 = \sum_s ([2^{v-2s}, 1^{2s}] + 2 \sum_t [2^{v-1-2s-t}, 1^{2s}]) \quad (4.51a)$$

$$\square''^2 = \sum_s ([2^{v-2s}, 1^{2s}] - 2 \sum_t (-1)^t [2^{v-1-2s-t}, 1^{2s}]) \quad (4.51b)$$

and

$$\square \square'' = \sum_s [2^{v-2s}, 1^{2s}]'' \quad (4.51c)$$

and hence to (Butler and Wybourne 1969)

$$\square_{\pm} \square_{\pm} = \sum_s ([2^{v-2s}, 1^{2s}]_{\pm} + \sum_t [2^{v-2-2s-2t}, 1^{2s}]) \quad (4.52a)$$

and

$$\square_{\pm} \square_{\mp} = \sum_s (\sum_t [2^{v-1-2s-2t}, 1^{2s}]) \quad (4.52b)$$

in the case of  $\Delta$ , (4.22a) leads to

$$\Delta^2 = [1^v] + 2 \sum_s [1^{v-1-s}] \quad (4.53a)$$

whilst for the difference irreps (Butler and Wybourne 1969)

we have

$$\Delta''^2 = [1^v] - 2 \sum_s (-1)^s [1^{v-1-s}] \quad (4.53b)$$

and

$$\Delta \Delta'' = \square'' = [1^v]'' \quad (4.53c)$$

From these results it is easy to rederive those of Brauer and Weyl (1935)

$$\Delta_{\pm} \Delta_{\pm} = [1^v]_{\pm} + \sum_s [1^{v-2-2s}] \quad (4.54a)$$

$$\Delta_{\pm} \Delta_{\mp} = \sum_s [1^{v-1-2s}] \quad (4.54b)$$

Furthermore, from (4.43) we have

$$\Delta \square = [\Delta; 1^v/Q] \quad (4.55a)$$

$$\Delta'' \square = [\Delta; 1^v/L]'' \quad (4.55b)$$

and in a similar manner

$$\Delta \square'' = \Delta^2 \Delta'' = \Delta'' [1^v/S] = \Delta'' [1^v/MQ] \quad (4.56a)$$

and

$$\Delta^{\square} = \Delta \Delta^2 = \Delta[1^{\nu}/R] = \Delta[1^{\nu}/LR] \quad (4.56b)$$

so from (4.43) we get

$$\Delta^{\square} = [\Delta; 1^{\nu}/Q]^{\square} = \sum_s [\Delta; 1^{\nu-s}] \quad (4.57a)$$

$$\Delta^{\square} = [\Delta; 1^{\nu}/L] = \sum_s (-1)^s [\Delta; 1^{\nu-s}] \quad (4.57b)$$

From these results it follows that

$$\Delta_{\pm}^{\square} = \sum_s [\Delta; 1^{\nu-2s}]_{\pm} \quad (4.58a)$$

$$\Delta_{\pm}^{\square} = \sum_s [\Delta; 1^{\nu-1-2s}]_{\mp} \quad (4.58b)$$

## II. Kronecker products of tensor irreps

There are four basic types of Kronecker products of the tensor irreps of  $SO_{2\nu}$  of the following forms:

$$[\lambda][\mu] \quad (p_{\lambda}, p_{\mu} < \nu) \quad (4.59a)$$

$$[\lambda][\mu]_{\pm} = \frac{1}{2}([\lambda][\mu] \pm [\lambda][\mu]') \quad (p_{\lambda} < \nu, p_{\mu} = \nu) \quad (4.59b)$$

$$[\lambda]_{\pm}[\mu]_{\pm} = \frac{1}{2}([\lambda][\mu] + [\lambda]''[\mu]'' \pm [\lambda][\mu]'' \pm [\lambda]''[\mu]) \quad (p_{\lambda}, p_{\mu} = \nu) \quad (4.59c)$$

$$[\lambda]_{\pm}[\mu]_{\mp} = \frac{1}{4}([\lambda][\mu] - [\lambda]''[\mu]'' \pm [\lambda]''[\mu] \mp [\lambda][\mu]') \quad (p_{\lambda}, p_{\mu} = \nu) \quad (4.59d)$$

The terms in  $[\lambda][\mu]$  may be found by use of (4.20) for  $O_{2\nu}$  followed by the use of the modification rule of (4.6a)

and the using the branching rules of  $O_{2V} \downarrow SO_{2V}$  (4.37a,b,c).

The terms in  $[\lambda]''[\mu]$  follow upon noting that

$$[\square; \lambda]''[\mu] = \sum_{\zeta} [\square; (\lambda/\zeta) \cdot (\mu/\zeta V)]'' \quad (4.60a)$$

while the terms in  $[\lambda]'' \cdot [\mu]''$  may be found by noting that

$$[\square; \lambda]''[\square; \mu]'' = \square'' \sum_{\zeta} [((\lambda/\zeta) \cdot (\mu/\zeta))/W] \quad (4.61)$$

where (4.51b) is used to evaluate the terms in  $\square''^2$ .

Thus to evaluate  $[21^4]_+[1^2]$  for  $SO_{10}$  we have from (4.59b).

$$[21^2]_+[1^2] = \frac{1}{2}([21^4][1^2] + [21^4]''[1^2]) \quad (4.62)$$

Use of (4.20) gives

$$\begin{aligned} [21^4][1^2] &= \sum_{\zeta} [(21^4/\zeta) \cdot (1^2/\zeta)] \\ &= [(21^4) \cdot (1^2)] + [(21^3) \cdot (1)] + [(1^5) \cdot (1)] \\ &\quad + [1^4] + [21^2] \\ &= [21^6] + [2^2 1^4] + [2^3 1^2] + [31^5] + [321^3] \\ &\quad + [31^3] + [2^2 1^2] + [21^4] + [1^6] + [21^4] \\ &\quad + [1^4] + [21^2] \end{aligned}$$

The application of the modification rule (4.6a) then gives

$$\begin{aligned} [21^4][1^2] &= [321^3] + 2[31^3] + [2^3 1^2] + 2[2^2 1^2] + 2[21^4] \\ &\quad + 2[21^2] + 2[1^4] \end{aligned} \quad (4.63)$$

We now have from (4.60)

$$\begin{aligned}
 [21^4]''[1^2] &= [\square; 1]''[1^2] = \sum_{\zeta} [\square; (1/\zeta) \cdot (1^2/\zeta V)]'' \\
 &= [\square; (1) \cdot (1^2/V)] + [\square; (0) \cdot (1/V)]'' \\
 &= [\square; (1) \cdot (1^2)]'' + [\square; 1]'' + [\square; 1]'' \\
 &= [\square; 21]'' + [\square; 1] + 2[\square; 1]'' \\
 &= [321^3]'' + [2^3 1^2]'' + 2[21^4]'' \quad (4.64)
 \end{aligned}$$

Inserting (4.63) and (4.64) into (4.62) gives

$$\begin{aligned}
 [21^4] + [1^2] &= \frac{1}{2}([321^3] + [321^3]') + [31^3] + \frac{1}{2}([2^3 1^2] + [2^3 1^2]') \\
 &\quad + [2^2 1^2] + [21^4] + [21^4]'' + [21^2] + [1^4] \\
 &= [321^3]_+ + [31^3] + [2^3 1^2]_+ + [2^2 1^2] + 2[21^4]_+ \\
 &\quad + [21^2] + [1^4]
 \end{aligned}$$

### III. Kronecker products of spin irreps

The product of two spin irreps necessarily yields a linear combination of tensor irreps. The two basic types of Kronecker products for spin irreps of  $SO_{2V}$  may be written as

$$\begin{aligned}
 [\Delta, \lambda]_{\pm} [\Delta; \mu]_{\pm} &= \frac{1}{4}([\Delta; \lambda][\Delta; \mu] + [\Delta; \lambda]''[\Delta; \mu]'' \\
 &\quad \pm [\Delta; \lambda]''[\Delta; \mu] \pm [\Delta; \lambda][\Delta; \mu]') \quad (4.65a)
 \end{aligned}$$

and

$$\begin{aligned}
[\Delta; \lambda]_{\pm} [\Delta; \mu]_{\mp} &= \frac{1}{4} ([\Delta; \lambda] [\Delta; \mu] - [\Delta; \lambda]'' [\Delta; \mu]'' \\
&\quad \pm [\Delta; \lambda]'' [\Delta; \mu] \mp [\Delta; \lambda] [\Delta; \mu]'')
\end{aligned} \tag{4.65b}$$

The individual terms appearing on the right-hand-sides of (4.65) may be evaluated as

$$\begin{aligned}
[\Delta; \lambda] [\Delta; \mu] &= \sum_{\zeta} [1^v; ((\lambda/\zeta) \cdot (\mu/\zeta))_{p_{\rho}} / Q] \\
&\quad + 2 \sum_{s=0}^{v-1-p_{\rho}} [1^{v-1-s}; ((\lambda/\zeta) \cdot (\mu/\zeta))_{p_{\rho}} / Q]
\end{aligned} \tag{4.66a}$$

$$\begin{aligned}
[\Delta; \lambda]'' [\Delta; \mu]'' &= \sum_{\zeta} ([1^v; ((\lambda/\zeta) \cdot (\mu/\zeta))_{p_{\rho}} / L] \\
&\quad - 2 \sum_{s=0}^{v-1-p_{\rho}} (-1)^s [1^{v-1-s}; ((\lambda/\zeta) \cdot (\mu/\zeta))_{p_{\rho}} / L])
\end{aligned} \tag{4.66b}$$

and

$$[\Delta; \lambda]'' [\Delta; \mu] = \sum_{\zeta} [1^v; (\lambda/\zeta Q) \cdot (\mu/\zeta L)]'' \tag{4.66c}$$

The subscript  $p_{\rho}$  in (4.66a) and (4.66b) is included as a reminder that the factors  $((\lambda/\zeta) \cdot (\mu/\zeta))_{p_{\rho}}$  appearing in (4.66a) and (4.66b) must, where necessary, be modified with the rules

$$(\rho) = (-1)^x (\rho-h) \quad \text{with } h = 2p_{\rho} - 2v - 1 \tag{4.67a}$$

and

$$(\rho) = (-1)^{x-1} (\rho-h) \quad \text{with } h = 2p_{\rho} - 2v - 1 \tag{4.67b}$$

respectively before dividing by  $Q$  and  $L$ . Likewise the

modification rules (4.50c) must be used in (4.66c) when required.

Consider the evaluation of  $[\Delta; 1^2]_+ \cdot [\Delta; 1]_\pm$  for  $SO_{10}$ .

From (4.66a) we have

$$\begin{aligned}
 [\Delta; 1^2][\Delta; 1] &= \sum_{\zeta} ([1^5; ((1^2/\zeta) \cdot (1/\zeta))_{p_\rho} / Q] \\
 &\quad + 2 \sum_{s=0}^{4-p_\rho} [1^{4-s}; ((1^2/\zeta) \cdot (1/\zeta))_{p_\rho} / Q] \\
 &= [1^5; (1^2 \cdot 1)/Q] + [1^5; (1 \cdot 0)/Q] \\
 &\quad + 2 \sum_{s=0}^1 [1^{4-s}; 1^3/Q] + 2 \sum_{s=0}^3 [1^{4-s}; 21/Q] \\
 &\quad + 2 \sum_{s=0}^3 [1^{4-s}; 1/Q] \\
 &= [321^3] + 2[321^2] + 2[321] + 2[32] + [31^4] \\
 &\quad + 2[31^3] + 2[31^2] + 2[31] + [2^3 1^2] + 2[2^3 1] \\
 &\quad + 2[2^3] + 2[2^2 1^3] + 4[2^2 1^2] + 4[2^2 1] + 2[2^2] \\
 &\quad + 3[21^4] + 6[21^3] + 6[21^2] + 4[21] + 2[2] \\
 &\quad + 2[1^5] + 4[1^4] + 4[1^3] + 2[1^2] + 2[1].
 \end{aligned}$$

and from (4.66b)



$$\begin{aligned}
[\Delta; 1^2]''[\Delta; 1]'' &= \sum_{\zeta} ([1^5; ((1^2/\zeta)(1/\zeta))_{p_{\rho}}/L] \\
&\quad - 2 \sum_{s=0}^{4-p_{\rho}} (-1)^s; ((1^2/\zeta)(1/\zeta))_{p_{\rho}}/L]) \\
&= [1^5; (1^2.1)/L] + [1^5; (1.0)/L] - 2 \sum_{s=0}^1 (-1)^s [1^{4-s}; 1^3/L] \\
&\quad - \sum_{s=0}^2 (-1)^s [1^{4-s}; 21/L] - 2 \sum_{s=0}^3 (-1)^s [1^{4-s}; 1/L] \\
&= [321^3] - 2[321^2] + 2[321] - 2[32] - [31^4] \\
&\quad + 2[31^3] - 2[31^2] + 2[31] + [3^21^2] - 2[2^31] \\
&\quad + 2[2^3] - 2[2^21^3] + 4[2^21^2] - 4[2^21] + 2[2^2] \\
&\quad + 3[21^4] - 6[21^3] + 6[21^2] - 4[21] + 2[2] - 2[1^5] \\
&\quad + 4[1^4] - 4[1^3] + 2[1^2] - 2[1]
\end{aligned}$$

and from (4.66c)

$$\begin{aligned}
[\Delta; 1^2]''[\Delta; 1] &= \sum_{\zeta} [1^5; (1^2/\zeta Q) \cdot (1/\zeta L)]'' \\
&= [1^5; (1^2/Q) \cdot (1/L)]'' + [1^5; (1/Q) \cdot (0/L)]'' \\
&= [1^5; (1^2+1+0) \cdot (1-0)]'' + [1^5; (1+0) \cdot (0)]'' \\
&= [1^5; 21 + 1^3 + 2 + 1]'' \\
&= [321^2]'' + [2^31^2]'' + [31^4]'' + [21^4]''.
\end{aligned}$$

and similarly

$$[\Delta; 1^2][\Delta; 1]^+ = [321^2]^+ + [2^3 1^2]^+ - [31^4]^+ + [21^4]^+$$

Using the above results in (4.65a) gives

$$\begin{aligned} [\Delta; 1^2]_+ [\Delta; 1]_+ &= \frac{1}{4}(2[321^3] + 2[321^3]^+ + 4[321] \\ &+ 4[31^3] + 4[31] + 2[2^3 1^2] + 2[2^3 1^2]^+ + 4[2^3] \\ &+ 8[2^2 1^2] + 4[2^2] + 6[21^4] + 2[21^4]^+ + 12[21^2] \\ &+ 4[2] + 8[1^4] + 4[1^2]) \\ &= [321^3]_+ + [321] + [31^3] + [31] + [2^3 1^2]_+ + [2^3] \\ &+ 2[2^2 1^2] + [2^2] + 2[21^4]_+ + [21^4]_- + 3[21^2] + [2] \\ &+ 2[1^4] + [1^2]. \end{aligned}$$

Likewise we may deduce that

$$\begin{aligned} [\Delta; 1^2]_+ [\Delta; 1]_- &= [321^2] + [32] + [31^4]_+ + [31^2] + [2^3 1] \\ &+ [2^2 1^3]_+ + [2^2 1^3]_- + 2[2^2 1] + 3[21^3] + 2[21] \\ &+ [1^5]_+ + [1^5]_- + 2[1^3] + [1] \end{aligned}$$

# VI. Kronecker products of tensor with spin irreps

The product of a tensor irrep with a spin irrep yields a linear combination of spin irreps.

Two distinct types of Kronecker products need to be considered for  $SO_{2v}$

$$[\Delta; \lambda]_{\pm} [\mu] \quad (4.68)$$

and

$$[\Delta; \lambda]_{\pm} [\square; \mu]_{\pm} \quad (4.69a)$$

$$[\Delta; \lambda]_{\pm} [\square; \mu]_{\mp} \quad (4.69b)$$

The first product (4.68) may be cast in the form

$$[\Delta; \lambda]_{\pm} [\mu] = \sum_{\zeta, s} [\Delta; (\lambda/\zeta) \cdot (\lambda/\zeta 1^s)]_{\pm} (-1)^s \quad (4.70)$$

with the modification rules in (4.50b) being used as appropriate.

The second products may be expanded as

$$\begin{aligned} [\Delta; \lambda]_{\pm} [\square; \mu]_{\pm} &= \frac{1}{4} ([\Delta; \lambda] [\square; \mu] + [\Delta; \lambda] " [\square; \mu] " \\ &\quad \pm [\Delta; \lambda] [\square; \mu] " \pm [\Delta; \lambda] " [\square; \mu] ) \end{aligned} \quad (4.71a)$$

$$\begin{aligned} [\Delta; \lambda]_{\pm} [\square; \mu]_{\mp} &= \frac{1}{4} ([\Delta; \lambda] [\square; \mu] - [\Delta; \lambda] " [\square; \mu] " \\ &\quad \pm [\Delta; \lambda] " [\square; \mu] \mp [\Delta; \lambda] [\square; \mu] ) \end{aligned} \quad (4.71b)$$

where  $[\Delta; \lambda] \cdot [\square; \mu]$  and  $[\Delta; \lambda]''[\square; \mu]$  are evaluated by replacing  $\mu$  by  $[\square; \mu]$  in

$$[\Delta; \lambda] \cdot [\mu] = \sum_{\zeta} [\Delta; (\lambda/\zeta) \cdot (\mu/\zeta Q)] \quad (4.72a)$$

$$[\Delta; \lambda]''[\mu] = \sum_{\zeta} [\Delta; (\lambda/\zeta) \cdot (\mu/\zeta L)] \quad (4.72b)$$

while

$$[\Delta; \lambda][\square; \mu]'' = \sum_{\zeta, s} [\Delta; (1^{v-s}/Q) \cdot ((\lambda/\zeta L) \cdot (\mu/\zeta) / 1^{s_P})] \quad (4.73a)$$

and

$$[\Delta; \lambda]''[\square; \mu]'' = \sum_{\zeta, s} [\Delta; (1^{v-s}/L) \cdot ((\lambda/\zeta Q) \cdot (\mu/\zeta)) / 1^{s_M}]'' \quad (4.73b)$$

We now have a complete prescription for evaluating any Kronecker product for  $SO_{2v}$ . The results are in many cases tedious. Practical calculations for  $SO_{2v}$  may be shortened by noting that if

$$[\lambda] [\mu] = g_{\lambda\mu}^v [v] \quad (4.74)$$

then

$$*([\lambda][\mu]) = g_{\lambda\mu}^v * [v] \quad (4.75)$$

where  $[\lambda]$  and  $[\mu]$  may be tensor or spin irreps. Thus for

$$[11]_+ [\Delta; 1]_- = [\Delta; 2]_+ + [\Delta; 1]_-$$

We immediately deduce that

$$[11]_- [\Delta; 1]_+ = [\Delta; 2]_- + [\Delta; 1]_+$$

#### 4.11 PLETHYSMS OF $SO_N$

The plethysms of irrep  $O_N$  follow from sec. 4.6

$$[\lambda] \otimes \{\mu\} = [(\{\lambda/C\} \otimes \{\mu\})/D] \quad (4.29a)$$

and

$$[\lambda]^\pm \otimes \{\mu\} = [(\{\lambda/C\} \otimes \{\mu\})/D]^{(\pm)w_\mu} \quad (4.29b)$$

and for the spin irreps for  $O_{2v}$

$$[\Delta; \lambda] \otimes \{\mu\} = \sum_{\zeta} [\Delta \otimes \{\zeta\}] [(\lambda/E) \otimes \{\mu \circ \zeta\})/D] \quad (4.30a)$$

and for  $O_{2v+1}$

$$[\Delta; \lambda] \otimes \{\mu\} = \sum_{\zeta} [\Delta \otimes \{\zeta\}] [(\{\lambda/E^\pm\} \otimes \{\mu \circ \zeta\})/D] \quad (4.30b)$$

where  $E^\pm = P^\pm C$ .

For the tensor irreps of  $SO_{2v+1}$  it suffices to use (4.29a). In the case of  $SO_{2v}$  if  $p_v < v$  we may also use (4.29a) provided (4.73) is used on the right-hand side and the modification rule (4.50a) used where necessary. For spin irreps of  $SO_{2v+1}$  (4.30b) suffices with the  $\pm$  removed to give (4.30a).

However, for the tensor irreps with  $p_\lambda = v$  and the spin irreps of  $SO_{2v}$  it is necessary to use difference characters and in particular

$$[\Delta; \lambda]'' \otimes \{\mu\} = \sum_{\zeta} [\Delta'' \otimes \{\zeta\}] [(\{\lambda/G\} \otimes \{\mu \circ \zeta\})/D] \quad (4.76a)$$

and

$$[\square; \lambda]'' \otimes \{\mu\} = \sum_{\zeta} [\square'' \otimes \{\zeta\}] [(\{\lambda/A\} \otimes \{\mu \circ \zeta\})/D] \quad (4.76b)$$

where use has been made of (4.43b) and (4.48b) and the S-function series identities of chapter 2.

It was worth pointing out that the automorphism  $*$  of  $SO_{2v}$  is such that

$$*[\Delta; \lambda]'' = -[\Delta; \lambda]'' \quad (4.77a)$$

and

$$*[\square; \lambda]'' = -[\square; \lambda]'' \quad (4.77b)$$

hence from (3.41)

$$\begin{aligned} [\Delta; \lambda]'' \otimes \{\tilde{\mu}\} &= (- *[\Delta; \lambda]'' ) \otimes \{\tilde{\mu}\} \\ &= *(-[\Delta; \lambda]'' \otimes \{\tilde{\mu}\}) \\ &= (-1)^{w_{\mu}} *([\Delta; \lambda]'' \otimes \{\mu\}) \end{aligned} \quad (4.78a)$$

and similarly

$$[\square; \lambda]'' \otimes \{\tilde{\mu}\} = (-1)^{w_{\mu}} *([\square; \lambda]'' \otimes \{\mu\}) \quad (4.78b)$$

whilst, more obviously

$$\begin{aligned} [\Delta; \lambda]_{\pm} \otimes \{\mu\} &= (*[\Delta; \lambda]_{\mp}) \otimes \{\mu\} \\ &= *([\Delta; \lambda]_{\mp} \otimes \{\mu\}) \end{aligned} \quad (4.79a)$$

and

$$\begin{aligned}
 [\square; \lambda]_{\pm} \otimes \{\mu\} &= (*[\square; \lambda]_{\mp}) \otimes \{\mu\} \\
 &= *([\square; \lambda]_{\mp} \otimes \{\mu\})
 \end{aligned} \tag{4.79b}$$

A knowledge of the plethysms involving  $[\Delta; \lambda]$ ,  $[\Delta; \lambda]''$ ,  $[\square; \lambda]$  and  $[\square; \lambda]''$  leads directly to a knowledge of the plethysms  $[\Delta; \lambda]_{\pm} \otimes \{\mu\}$  and  $[\square; \lambda]_{\pm} \otimes \{\mu\}$ . Let  $[\lambda]_{\pm}$  stand for  $[\Delta; \lambda]_{\pm}$  then

$$(2[\lambda]_{\pm}) \otimes \{\mu\} = \sum_{\zeta} ([\lambda]_{\pm} \otimes \{\mu/\zeta\}) ([\lambda]_{\pm} \otimes \{\zeta\}) \tag{4.80}$$

Application of (4.41) then leads to

$$(2[\lambda]_{+}) \otimes \{\mu\} = \sum_{\zeta} ([\lambda] \otimes \{\mu/\zeta\}) ([\lambda]'' \otimes \{\zeta\}) \tag{4.81a}$$

and

$$(2[\lambda]_{-}) \otimes \{\mu\} = \sum_{\zeta} (-1)^{w_{\zeta}} ([\lambda] \otimes \{\mu/\zeta\}) ([\lambda]'' \otimes \{\tilde{\zeta}\}) \tag{4.81b}$$

where the first of (4.81a) and (4.81b) include  $2([\lambda]_{+} \otimes \{\mu\})$  and  $2([\lambda]_{-} \otimes \{\mu\})$  respectively. Thus, for example, we find

$$\begin{aligned}
 [\lambda]_{+} \otimes \{21\} &= \frac{1}{2}([\lambda] + [\lambda]'') \otimes \{21\} - [\lambda]_{+}^3 \\
 &= \frac{1}{2}([\lambda] \otimes \{21\} + [\lambda]'' \otimes \{21\}) \\
 &\quad - \frac{1}{4} [\lambda]_{-} ([\lambda]^2 + [\lambda]''^2)
 \end{aligned}$$

Continuing in this manner we may express any plethysm for

$SO_{2v}$  in terms of plethysma and products of  $[\lambda]$  and  $[\lambda]''$ . The evaluation of the plethysms in  $[\lambda]$  and  $[\lambda]''$  requires the resolution of the Kronecker powers of the basic irreps with character  $\Delta$ ,  $\Delta''$  and  $\square$ . In the latter case we have

$$\square \otimes \{\mu\} = (\Delta \Delta'') \otimes \{\mu\} = \sum_{\zeta} (\Delta \otimes \{\zeta\} (\Delta'' \otimes \{\mu \circ \zeta\})) \quad (4.82a)$$

Incidentally,

$$\square \otimes \{\mu\} = [1^v] \otimes \{\mu\} = [([1^v] \otimes \{\mu\})/D] \quad (4.82b)$$

and thus our task reduces to evaluating the basic spin plethysms  $\Delta \otimes \{\mu\}$  and  $\Delta'' \otimes \{\mu\}$ .

#### 4.12 SYMMETRIZED KRONECKER SQUARE AND THE CLASSIFICATION OF IRREPS OF $SO_N$

In sec. 4.7 we have given the plethysm of  $O_N$ . For  $SO_N$ , in the spin case of  $SO_{2v+1}$  it is only necessary to delete the  $\sharp$  in (4.3) and (4.4), which then yield identical formulae for  $v$  even and odd

$$\Delta \otimes \{2\} = [1^v] + \sum_x ([1^{v-3-4x}] + [1^{v-4-4x}]) \quad (4.83a)$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{v-1-4x}] + [1^{v-2-4x}]) \quad (4.83b)$$

For  $SO_{2v}$  the same procedure yields



$$\Delta \otimes \{2\} = [1^v] + \sum_x ([1^{v-1-4x}] + [1^{v-3-4x}] + 2[1^{v-4-4x}]) \quad (4.84a)$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{v-1-4x}] + 2[1^{v-2-4x}] + [1^{v-3-4x}]) \quad (4.84b)$$

In order to cope with the  $\Delta''$ , it is helpful to note first that

$$\Delta \otimes \{1^2\} = (\Delta_+ + \Delta_-) \otimes \{1^2\} = \Delta_+ \otimes \{1^2\} + \Delta_- \otimes \{1^2\} + \Delta_+ \Delta_-$$

So from (4.84b) and (4.54b) we have

$$\Delta_+ \otimes \{1^2\} + \Delta_- \otimes \{1^2\} = 2 \sum_x [1^{v-2-4x}]$$

Each term of the right-hand side is invariant under the automorphism  $*$ . This implies that the same is true for both on the left hand side. Hence

$$\Delta_+ \otimes \{1^2\} = *(\Delta_+ \otimes \{1^2\}) = \Delta_- \otimes \{1^2\}$$

Therefore

$$\begin{aligned} \Delta'' \otimes \{2\} &= (\Delta_+ - \Delta_-) \otimes \{2\} = \Delta_+ \otimes \{2\} - \Delta_+ \Delta_- + \Delta_- \otimes \{1^2\} \\ &= \Delta_+^2 - \Delta_+ \Delta_- = \Delta_+ \Delta'' \end{aligned}$$

leading to

$$\Delta'' \otimes \{2\} = (\Delta''^2 + \square'')/2$$

and likewise

$$\Delta'' \otimes \{1^2\} = (\Delta''^2 - \square'')/2$$

with the terms in (4.53b). Hence (Butler and Wybourne 1969)

$$\Delta'' \otimes \{2\} = [1^v]_+ - \sum_x (-1)^x [1^{v-t-x}] \quad , \quad (4.94a)$$

$$\Delta'' \otimes \{1^2\} = [1^v]_- - \sum_x (-1)^x [1^{v-1-x}] \quad . \quad (4.94b)$$

and combining this result with (4.84) yields

$$\Delta_{\pm} \otimes \{2\} = \{1^v\}_{\pm} + \sum_x [1^{v-4-4x}] \quad (4.95a)$$

$$\Delta_{\pm} \otimes \{1^2\} = \sum_x [1^{v-2-4x}] \quad (4.95b)$$

in conformity with the results of Littlewood (1947). For the tensor irreps for  $SO_{2v+1}$  and  $SO_{2v}$  with  $p_{\lambda} < v$  we have

$$[\lambda] \otimes \{\mu\} = \sum_{\zeta} \{\lambda/\zeta\} \otimes \{\mu\} \quad \mu = (2) \text{ or } (1^2) \quad (4.96)$$

However in  $p_{\lambda} = v$  for  $SO_{2v}$  we have to consider the difference irreps. Here we only consider the case of  $\square$  the resolution of  $\square^2$  follow from the plethysm (Littlewood 1943).

$$\square \otimes \{2\} = \{1^2\} \otimes \{2\} = \sum_{s,t} [2^{v-s-6}, 1^{2s}] \quad (4.97)$$

and the use of the modification rule (4.50a); whilst the resolution of  $\square''^2$  proceeds by writing

$$\begin{aligned}
\Box'' \otimes \{2\} &= (\Delta\Delta'') \otimes \{2\} = (\Delta \otimes \{2\})(\Delta'' \otimes \{2\}) \\
&+ (\Delta \otimes \{1^2\})(\Delta'' \otimes \{1^2\})
\end{aligned} \tag{4.98}$$

leading to the results

$$\begin{aligned}
\Box \otimes \{2\} &= \sum_s \{[2^{s-4}, 1^{4s}] + \sum_t ([2^{\nu-1-4s-2t}, 1^{4s}] \\
&+ 2[2^{\nu-2-4s-2t}, 1^{4s}] + [2^{\nu-3-4s-2t}, 1^{2+4s}])\}
\end{aligned} \tag{4.99a}$$

$$\begin{aligned}
\Box \otimes \{1^2\} &= \sum_s \{[2^{\nu-2-4s}, 1^{2+2s}] + \sum_t ([2^{\nu-1-4s-2t}, 1^{4s}] \\
&+ [2^{\nu-3-4s-2t}, 1^{2+4s}] + 2[2^{\nu-4-4s-2t}, 1^{2+4s}])\}
\end{aligned} \tag{4.99b}$$

$$\Box'' \otimes \{2\} = \sum_s \{[2^{\nu-2s}, 1^{2s}]_{+(-1)^s} - \sum_t (-1)^t [2^{\nu-1-2s-t}, 1^{2s}]\} \tag{4.99c}$$

$$\Box'' \otimes \{1^2\} = \sum_s \{[2^{\nu-2s}, 1^{2s}]_{-(-1)^s} - \sum_t (-1)^t [2^{\nu-1-2s-t}, 1^{2s}]\} \tag{4.99d}$$

Hence

$$\Box_{\pm} \otimes \{2\} = \sum_s \{[2^{\nu-4s}, 1^{4s}]_{\pm} + \sum_t [2^{\nu-2-4s-2t}, 1^{4s}]\} \tag{4.100a}$$

$$\Box_{\pm} \otimes \{1^2\} = \sum_s \{[2^{\nu-2-4s}, 1^{2+4s}]_{\pm} + \sum_t [2^{\nu-4-4s-2t}, 1^{2+4s}]\} \tag{4.100b}$$

Returning to the classification of  $SO_N$  irreps we distinguish  $N = 2\nu+1$  and  $N = 2\nu$ . For  $N = 2\nu+1$  the tensor irreps  $[\lambda]$  remain all orthogonal while the spin irreps  $[\Delta; \lambda]$  are orthogonal if  $N = 1, 7 \pmod{8}$  and symplectic if  $N = 3, 5 \pmod{8}$ .

For  $SO_{2v}$  the tensor irreps  $[\lambda]$  are orthogonal if  $p_\lambda < v$ . If  $p_\lambda = v$  then  $[\lambda]_\pm$  is orthogonal if  $N = 0 \pmod{4}$  and otherwise complex. The spin irreps  $[\Delta; \lambda]_\pm$  are orthogonal if  $N = 0 \pmod{8}$ , complex if  $N = 2, 6 \pmod{8}$  and symplectic if  $N = 4 \pmod{8}$ .

#### 4.13 RESOLUTION OF THE BASIC SPIN KRONECKER CUBES OF $SO_N$

The evaluation of the plethysms of  $\Delta \otimes \{\mu\}$  for the groups  $O_{2v+1}$  or  $O_{2v}$  is equivalent, by virtue of (3.62), to determining the reduction of the irrep  $\{\mu\}$  of  $O_{2v}$  into irreps of  $O_{2v+1}$  or  $O_{2v}$  respectively. The dimension of the irreps are readily found (sec. 3.3 and sec. 4.3). In particular

$$D_{2v}\{3\} = 2^v(2^v+1)(2^v+2)/6 \quad (4.101a)$$

$$D_{2v}\{21\} = 2^v(2^v+1)(2^v-1)/6 \quad (4.101b)$$

$$D_{2v}\{1^3\} = 2^v(2^v-1)(2^v-2)/6 = \binom{2^v}{3} \quad (4.101c)$$

whilst

$$D_{2v+1}[1^x] = \binom{2^{v+1}}{x} \quad (4.102a)$$

$$D_{2v}[1^x] = \binom{2^v}{x} \quad (4.102b)$$

and of course

$$D_{2v+1}[\Delta] = D_{2v}[\Delta] = 2^v \quad (4.103a)$$

$$D_{2v}[\Delta''] = D_{2v}[\square''] = 0 \quad (4.103b)$$

$$D_{2v}[\Delta_{\pm}] = 2^{v-1} \quad (4.103c)$$

In evaluating the resolved Kronecker cube of  $\Delta$ , we endeavour to express the results as a product of the basic spin irrep  $\Delta$  and a series of irreps of  $O_N$  of the generic type  $[1^X]$ . It is a non-trivial task to distinguish between mutually associated pairs of irreps, so that from now on we limit attention to the groups  $SO_N$ . In the case of  $N = 2v+1$  and  $N = 2v$ , with  $v = 1, 2, \dots$ , the results for  $\Delta \otimes \{21\}$  may be readily evaluated, exploiting known isomorphisms (automorphisms in the case  $n = 8$ ) and explicit evaluation using Kronecker products and dimension checks (computer generated tables in the case  $n = 10$  by courtesy of Dr P.H. Butler) together with the branching rules

$$\begin{array}{lll}
 U_{2v} \downarrow SO_{2v+1} \downarrow SO_{2v} & & \\
 [0] & [0] & \\
 [1] & [1]+[0] & \\
 [1^X] & [1^X]+[1^{X-1}] & \\
 \{1\} \quad \Delta & \Delta_+ + \Delta_- & \\
 \{21\} \quad \Delta \otimes \{21\} & (\Delta_+ + \Delta_-) \otimes \{21\} & 
 \end{array} \quad (4.104)$$

These results for  $SO_{2v+1}$  with  $v = 1, 2, \dots, 5$  are all of the form

$$\Delta \otimes \{21\} = \Delta([1^{v-1}] + [1^{v-4}] + \dots)$$

which suggests that in general

$$\Delta \otimes \{21\} = \Delta \sum_x [1^{v-1-3x}] \quad (4.105)$$

This may be checked dimensionally, and is in agreement with the combinatorial identity (Breach 1980)

$$(2^{v+1})(e^v - 1)/3 = \sum_x \binom{2v+1}{v-1-3x} \quad (4.106)$$

Analogous identities can be established for the terms in  $\Delta \otimes \{3\}$  and  $\Delta \otimes \{1^3\}$ . Alternatively, and more simply, the use of (3.52) gives the complete results for  $SO_{2v+1}$ :

$$\begin{aligned} \Delta \otimes \{3\} = \Delta \sum_x ([1^{v-12x}] - [1^{v-1-12x}] + [1^{v-3-12x}] \\ + [1^{v-8-12x}] - [1^{v-10-12x}] + [1^{v-11-12x}]) \end{aligned} \quad (4.107a)$$

and

$$\begin{aligned} \Delta \otimes \{1^3\} = \Delta \sum_x ([1^{v-2-12x}] - [1^{v-4-12x}] + [1^{v-5-12x}] \\ + [1^{v-6-12x}] - [1^{v-7-12x}] + [1^{v-9-12x}]) \end{aligned} \quad (4.107b)$$

The corresponding results for  $SO_{2v}$  follow from the branching rules (4.104) which give

$$\begin{aligned} \Delta \otimes \{3\} = \Delta \{[1^v] + \sum_x (-[1^{v-2-12x}] + [1^{v-3-12x}] + [1^{v-4-12x}] \\ + [1^{v-8-12x}] + [1^{v-9-12x}] - [1^{v-10-12x}] + 2[1^{v-12-12x}]) \end{aligned} \quad (4.108a)$$

$$\Delta \otimes \{21\} = \Delta \sum_{\mathbf{x}} ([1^{\nu-1-3\mathbf{x}}] + [1^{\nu-2-3\mathbf{x}}]) \quad (4.108b)$$

$$\begin{aligned} \Delta \otimes \{1^3\} = \Delta \sum_{\mathbf{x}} ([1^{\nu-2-12\mathbf{x}}] + [1^{\nu-3-12\mathbf{x}}] - [1^{\nu-4-12\mathbf{x}}] \\ + 2[1^{\nu-6-12\mathbf{x}}] - [1^{\nu-8-12\mathbf{x}}] + [1^{\nu-9-12\mathbf{x}}] \\ + [1^{\nu-1\nu-12\mathbf{x}}]) \end{aligned} \quad (4.108c)$$

It only remains to prove the one result upon which (4.107) and (4.108) depend, namely (4.105). This may be done by induction with respect to  $\nu$  and the use of branching rules (King 1975b).

$$\begin{array}{ccc} U_{2\nu} & \downarrow & SO_{2\nu+1} & \downarrow & SO_{2\nu-1} \\ & & [1] & & [1] + 2[0] \\ & & [\lambda] & & [\lambda/MM]' \\ \{1\} & \Delta & & & 2\Delta' \\ \{21\} & \Delta \otimes \{21\} & & & (2\Delta') \otimes \{21\} \end{array} \quad (4.109)$$

where a prime is used to distinguish irreps of  $SO_{2\nu-1}$  from those of  $SO_{2\nu+1}$ . Then we have the branching rule of  $SO_{2\nu+1} \downarrow SO_{2\nu-1}$

$$\Delta \otimes \{21\} \downarrow (2\Delta') \otimes \{21\} = 2(\Delta' \otimes \{21\} + \Delta'^3) \quad (4.110)$$

writing

$$\Delta \otimes \{21\} = \Delta[X_\nu] \quad (4.111a)$$

and

$$\Delta' \otimes \{21\} = \Delta' [X'_{v-1}] \quad (4.111b)$$

this yields

$$[X_v/MM] = [X'_{v-1} + \Delta'^2]$$

and hence the recurrence relation

$$[X_v] = [X'_{v-1} + \Delta'^2]/LL$$

Assuming as the basis of an induction argument that

$$[X'_{v-1}] = \sum_x [1^{v-2-3x}],$$

and taking

$$\Delta'^2 = \sum_x [1^{v-1-3x}],$$

It is straightforward to show that

$$[X_v] = \sum_x [1^{v-1-3x}] \quad (4.112)$$

as required. The induction argument is then completed by means of the known validity of (4.105) in the cases  $v=1$  and  $v=2$ , thus proving the general validity of (4.105).

Having proved in this way the validity of not only (4.105) but also (4.107) and (4.108), it is then necessary to consider the Kronecker cubes of  $\Delta''$ . Once again explicit



results were obtained for  $SO_{2v}$  with  $v = 2, 3, 4$  and  $5$ , giving

$$\begin{aligned}
 SO_4 & \quad -\Delta''([1] - [0]) \\
 SO_6 & \quad -\Delta''([1^2] - [1]) \\
 SO_8 & \quad -\Delta''([1^3] - [1^2] - [0]) \\
 SO_{10} & \quad -\Delta''([1^4] - [1^3] - [1] + [0])
 \end{aligned} \tag{4.113}$$

where, remarkably, the bracketed terms are of total dimension  $3^{v-1}$  in each case. The general series is then identified as

$$\begin{aligned}
 \Delta'' \times \{21\} = \Delta'' \sum_x & (-[1^{v-t-6x}] + [1^{v-2-6x}] + [1^{v-4-6x}] \\
 & - [1^{v-5-6x}])
 \end{aligned} \tag{4.114}$$

by virtue of the combinatorial identity (Breach 1980)

$$3^{v-1} = \sum_x \left\{ \binom{2v}{v-1-6x} - \binom{2v}{v-2-6x} - \binom{2v}{v-4-6x} + \binom{2v}{v-5-6x} \right\} \tag{4.115}$$

Then we have

$$\Delta'' \otimes \{3\} = \Delta'' \{[1^v]_+ - \sum_x (-1)^x [1^{v-3-3x}]\} \tag{4.116a}$$

and

$$\Delta'' \otimes \{1^3\} = \Delta'' \{[1^v]_- - \sum_x (-1)^x [1^{v-3-2x}]\} \tag{4.116b}$$

Finally we find

$$\begin{aligned} \Delta_{\pm} \otimes \{3\} &= \Delta_{\pm}[1^v]_{\pm} - \sum \{-\Delta_{\pm}[1^v]_{\pm} + \Delta_{\pm}[1^{v-3-12x}] \\ &+ \Delta_{\pm}[1^{v-9-12x}] - \Delta_{\pm}[1^{v-10-12x}] + \Delta_{\pm}[1^{v-12-12x}]\} \quad (4.117a) \end{aligned}$$

$$\Delta_{\pm} \otimes \{21\} = \sum_x \{\Delta_{\pm}[1^{v-2-6x}] - \Delta_{\mp}[1^{v-3-6x}] + \Delta_{\pm}[1^{v-4-6x}]\} \quad (4.117b)$$

$$\begin{aligned} \Delta_{\pm} \otimes \{1^3\} &= \sum_x \{\Delta_{\mp}[1^{v-3-12x}] - \Delta_{\pm}[1^{v-4-12x}] \\ &+ \Delta_{\pm}[1^{v-6-12x}] - \Delta_{\pm}[1^{v-8-12x}] + \Delta_{\mp}[1^{v-9-12x}]\} \quad (4.117c) \end{aligned}$$

It is now straightforward to resolve  $\square^3$  making use of (4.82).

We list the plethysms for partitions ( $\mu$ ) of weight three or less for  $SO_{10}$  in table IV.

#### 4.14 RESOLUTION OF THE BASIC SPIN KRONECKER FOURTH POWERS OF $SO_{2v}$

If the second and third Kronecker powers of  $\lambda$  are known, then as pointed out in sec. 3.9, chapter 3, the fourth power of  $\lambda$  becomes completely resolvable.

For the basic spin irrep of  $SO_{2v}$  we have obtained the results for  $\Delta_{\pm} \otimes \{3\}$ ,  $\Delta_{\pm} \otimes \{21\}$  and  $\Delta_{\pm} \otimes \{1^3\}$  in sec. 4.14.

In order to resolve the fourth power of  $\Delta_{\pm}$  it is only necessary to evaluate the Kronecker products of  $[1^x][1^{x'}]$  and plethysms of the type  $[1^x] \otimes \{2\}$  and  $[1^x] \otimes \{1^2\}$ . In fact, we have

$$\begin{aligned} [1^x][1^{x'}] &= \sum_{\zeta} [(1^x/\zeta)(1^{x'}/\zeta)] \\ &= \sum_{q=0}^{\min(x, x')} [(1^{x-q})(1^{x'-q})] \quad (4.118) \end{aligned}$$

TABLE IV: Plethysms  $\square'' \otimes \{\lambda\}$  for  $SO_{10}$ 

$$\square'' \otimes \{2\} = ([2] + [21^2] + [2^3] + [21^4]_+ + [2^3 1^2]_- + [2^5]_+)$$

$$- ([0] + [1^2] + [1^4] + [2^2] + [2^2 1^2] + [2^4])$$

$$\square'' \otimes \{3\} = \square''(( [1^2] + [21^4]_+ + [2^3] + [2^5]_+ ) - ([0] + [1^4]$$

$$+ [2^2] + [2^3 1^2]_+))$$

$$\square'' \otimes \{21\} = \square''(( [2] + [21^2] + [2^3 1^2]_+ + [2^3 1^2]_- )$$

$$- (2[1^2] + [2^2 1^2] + [2^4]))$$

and

$$[1^x] \otimes \{2\} = \sum_{s,t} [2^{x-2s-t}, 1^{4s}] \quad (4.119a)$$

$$[1^x] \otimes \{1^2\} = \sum_{s,t} [2^{x-1-2s-t}, 1^{2+4s}] \quad (4.119b)$$

In view of the current interest in  $SO_{10}$  grand unified theory, we give in Table V the resolution of the  $\Delta_+ \otimes \{\mu\}$  irrep of  $SO_{10}$  for all partitions  $(\mu)$  rank four or less.

TABLE V: plethysms  $\Delta_+ \otimes \{\lambda\}$  for  $SO_{10}$ 

$$\begin{aligned}
\Delta_+ \otimes \{1\} &= [\Delta; 0]_+ \\
\Delta_+ \otimes \{2\} &= [1] + [1^5]_+ \\
\Delta_+ \otimes \{1^2\} &= [1^3] \\
\Delta_+ \otimes \{3\} &= [\Delta; 1]_+ + [\Delta; 1^5]_+ \\
\Delta_+ \otimes \{31\} &= [\Delta; 0]_- + [\Delta; 1]_+ + [\Delta; 1^3]_+ \\
\Delta_+ \otimes \{1^3\} &= [\Delta; 1^2]_- \\
\Delta_+ \otimes \{4\} &= [2] + [21^4]_+ + [2^5]_+ \\
\Delta_+ \otimes \{31\} &= [1^2] + [1^4] + [21^2] + [21^4]_+ + [2^3 1^2]_+ \\
\Delta_+ \otimes \{2^2\} &= [0] + [1^4] + [2] + [21^4]_+ + [2^3] \\
\Delta_+ \otimes \{21^2\} &= [1^2] + [1^4] + [21^4]_- + [2^2 1^2] \\
\Delta_+ \otimes \{1^4\} &= [21^2] + [2^2]
\end{aligned}$$

## CHAPTER 5

### THE REPRESENTATIONS OF SYMMETRIC GROUP $S_n$

#### - THE METHOD OF REDUCED NOTATION

#### 5.1 AN n-INDEPENDENT REDUCED NOTATION FOR THE ORDINARY IRREPS OF $S_n$

The possibility of developing an essentially n-independent resolution of Kronecker product was first considered by Murnaghan (1937, 1938), who suggested the use of a reduced notation for labelling the irreps of  $S_n$  that is n-independent.

The systematic evaluation of the inner products  $\{\lambda\} \circ \{\mu\}$  of S-functions was outlined in Sec. 3.5. Using those methods we have readily found that

$$\{n-1, 1\} \circ \{n-1, 1\} = \{n-2, 1^2\} + \{n-2, 2\} + \{n-1, 1\} + \{n\} \quad (5.1)$$

which holds for all n if for small n any non-standard S-functions have to be standardized as in sec. 2.3. Thus we have

$$\{31\} \circ \{31\} = \{21^2\} + \{2^2\} + \{31\} + \{4\}$$

$$\{41\} \circ \{41\} = \{31^2\} + \{32\} + \{41\} + \{5\}$$

$$\{51\} \circ \{41\} = \{41^2\} + \{42\} + \{51\} + \{6\}$$

while

$$\begin{aligned} \{21\} \circ \{21\} &= \{1^3\} + \{1, 2\} + \{2, 1\} + \{3\} \\ &= \{1^3\} + \{21\} + \{3\} \end{aligned}$$

since  $\{1,2\} = -\{1,2\} = 0$  by (2.31a) and the above observations suggest the possibility of developing an essentially  $n$ -independent resolution of Kronecker products for  $S_n$  using a reduced notation for labelling the irreps of  $S_n$  in an  $n$ -independent manner. This stratagem allows us to escape the limitations of tables of characters of  $S_n$ .

In the reduced notation, the irrep of  $S_n$  usually labelled by the symbol  $[\lambda] \equiv [n-w_\mu, (\mu_1, \mu_2, \dots)]$ , with  $(\mu)$  being a partition of  $w_\mu$ , is labelled by the symbol  $\langle \mu \rangle \equiv \langle \mu_1, \mu_2, \dots \rangle$ . In the reduced notation (5.1) assumes the  $n$ -independent form

$$\langle 1 \rangle \langle 1 \rangle = \langle 1^2 \rangle + \langle 1 \rangle + \langle 0 \rangle \quad (5.2)$$

In the reduced notation the two ordinary one-dimensional irreps  $[n]$  and  $[1^n]$  become labelled  $\langle 0 \rangle$  and  $\langle 1^{n-1} \rangle$ . The  $\langle 1^{n-1} \rangle$  irrep will frequently be designated as  $\langle \tilde{0} \rangle$  where the tilde reminds us that it is formed from the association of  $\langle 0 \rangle$  and define

$$\langle \tilde{\mu} \rangle \equiv \langle \tilde{0} \rangle \langle \mu \rangle \quad (5.3)$$

Care must be exercised in interpreting  $\langle \tilde{\mu} \rangle$  in the reduced notation, as the tilde operation does not imply conjugation of the partition  $(\mu)$  as it does in the standard notation. Thus  $\langle 1 \rangle \leftrightarrow [n-1, 1]$  while  $\langle \tilde{1} \rangle \leftrightarrow [\widetilde{n-1, 1}] \equiv [21^{n-2}] \equiv \langle 1^{n-2} \rangle$  generally we have  $\langle \tilde{1}^x \rangle = \langle 1^{n-1-x} \rangle$ .

In transforming from the reduced notation  $\langle \mu \rangle$  to the  $N$ -dependent standard  $[\lambda] \equiv [n-w_\mu, (\mu)]$  the resulting symbols may not be in the standard form. However, non-standard

symbols can always be reordered to give the standard form by use of (2.31).

## 5.2 THE SYMMETRIC GROUP $S_n$ AS A SUBGROUP OF $O_n$

The ordinary representations of  $S_n$  may be given an orthogonal Young-Yamanouchi realization (Hamermesh 1962) and hence the ordinary irreps of  $S_n$  are all of the orthogonal type, though not necessarily unimodular. As a consequence the symmetric group  $S_n$  may be regarded as a subgroup of the full orthogonal group  $O_n$  i.e.

$$O_n \supset S_n \quad (5.4)$$

Here we choose the embedding that leads to decomposition of the vector irrep  $[1]$  of  $O_n$  as

$$O_n \downarrow S_n: [1] \downarrow [n-1, 1] + [n] \quad (5.5a)$$

Since for  $O_n$  we have

$$[\lambda][\tilde{0}] = [\tilde{\lambda}]$$

we have

$$\begin{aligned} [\tilde{1}] \downarrow [\widetilde{n-1, 1}] + [\tilde{n}] \\ = [21^{n-2}] + [1^n] \end{aligned} \quad (5.5b)$$



Other embeddings are possible but (5.5) will suffice for our purposes.

In reduced notation (5.5) will be written as

$$O_n \downarrow S_n \quad [1] \downarrow \langle 1 \rangle + \langle 0 \rangle \quad (5.6)$$

The decomposition of an arbitrary tensor irrep  $[\lambda]$  of  $O_n$  into  $S_n$  irreps will yield the terms contained in the plethysm

$$\begin{aligned} (\langle 1 \rangle + \langle 0 \rangle) \otimes [\lambda] \\ = \langle 1 \rangle \otimes \{\lambda/G\} \end{aligned} \quad (5.7)$$

Plethysms of the type

$$\langle 1 \rangle \otimes \{\mu\} \quad (5.8)$$

may be evaluated by noting that (3.80) in reduced notation corresponds to

$$\langle 1 \rangle \otimes \{1^r\} = \langle 1^r \rangle \quad (5.9)$$

The S-function  $\{\mu\}$  may be expanded as a sum of products of elementary symmetric functions  $a_r = \{1^r\}$  via (2.29) to reduce the evaluation of (5.8) to the sum of products of all of the irreps of type  $\langle 1^X \rangle$ . The evaluation of products of  $\langle 1^X \rangle$  will be given later.

Thus to evaluate

$$(\langle 1 \rangle + \langle 0 \rangle) \otimes [21]$$

we have from (5.7)

$$(\langle 1 \rangle + \langle 0 \rangle) \otimes [21]$$

$$= \langle 1 \rangle \otimes (\{21\} + \{1^2\} - \{2\} - \{0\})$$

then

$$\langle 1 \rangle \otimes \{21\} = \langle 1 \rangle \otimes (\{1^2\}\{1\} - \{1^3\})$$

$$= \langle 1^2 \rangle \langle 1 \rangle - \langle 1^3 \rangle$$

$$= \langle 21 \rangle + \langle 1^2 \rangle + \langle 2 \rangle + \langle 1 \rangle$$

leading to

$$O_n \downarrow S_n \quad [21] \downarrow \langle 21 \rangle + 2\langle 20 \rangle + 2\langle 1^2 \rangle + 2\langle 1 \rangle$$

The above prescription, while tedious in application, is complete and capable of being programmed for computer evaluation. As an example we list the  $O_n \downarrow S_n$  branching rules for partitions of four or less in table VI.

### 5.3 REDUCED NOTATION FOR SPIN IRREPS OF $S_n$

It is readily seen that under  $O_n \downarrow S_n$  we have for the basic spin irrep  $\Delta$  of  $O_n$

TABLE VI: Branching rules  $O_n \downarrow S_n$ 

$O_n$	$S_n$
[0]	$\langle 0 \rangle$
[1]	$\langle 1 \rangle + \langle 0 \rangle$
[1 <sup>2</sup> ]	$\langle 1^2 \rangle + \langle 1 \rangle$
[2]	$\langle 2 \rangle + 2\langle 1 \rangle + \langle 0 \rangle$
[1 <sup>3</sup> ]	$\langle 1^3 \rangle + \langle 1^2 \rangle$
[21]	$\langle 21 \rangle + 2\langle 2 \rangle + 2\langle 1^2 \rangle + 2\langle 1 \rangle$
[3]	$\langle 3 \rangle + 2\langle 2 \rangle + \langle 1^2 \rangle + 3\langle 1 \rangle + 2\langle 0 \rangle$
[1 <sup>4</sup> ]	$\langle 1^4 \rangle + \langle 1^3 \rangle$
[21 <sup>2</sup> ]	$\langle 21^2 \rangle + 2\langle 21 \rangle + 2\langle 1^3 \rangle + \langle 2 \rangle + 2\langle 1^2 \rangle$
[2 <sup>2</sup> ]	$\langle 2^2 \rangle + 2\langle 21 \rangle + \langle 3 \rangle + 3\langle 2 \rangle + \langle 1^2 \rangle + \langle 1 \rangle$
[31]	$\langle 31 \rangle + 3\langle 21 \rangle + 2\langle 3 \rangle + \langle 1^3 \rangle + 4\langle 2 \rangle + 5\langle 1^2 \rangle + 4\langle 1 \rangle + \langle 0 \rangle$
[4]	$\langle 4 \rangle + \langle 21 \rangle + 2\langle 3 \rangle + 4\langle 2 \rangle + 2\langle 1^2 \rangle + 5\langle 1 \rangle + 3\langle 0 \rangle$

$$O_n \downarrow S_n \quad \Delta \downarrow [n], {}^\dagger \quad (5.10)$$

a result that holds for all  $n$ . This suggests that it should be possible to develop an  $n$ -independent reduced notation for the spin irreps as well as for ordinary irreps of  $S_n$ , e.g.,

$$\Delta \downarrow \langle 0 \rangle, {}^\dagger$$

Since for  $O_n$

$$[\Delta; \lambda] = \Delta[\lambda/P]$$

and noting (5.6) and (5.7) together with the  $S$ -function series relation  $PG = A$  we have under

$$O_n \downarrow S_n \quad [\Delta; \lambda] \downarrow \langle 0 \rangle, {}^\dagger [\langle 1 \rangle \otimes \{\lambda/A\}] \quad (5.11)$$

where the plethysm involves ordinary irreps of  $S_n$  and is evaluated as in previous section. The above result justifies introducing an  $n$ -independent reduced notation for spin irreps of  $S_n$  by writing

$$\langle \mu \rangle' = [n - w_\mu, (\mu)]' \quad (5.12)$$

A short list of the branching rules of (5.11) is given in table VII.

TABLE VII:  $[\Delta; 0] \downarrow \langle 0 \rangle, {}^\dagger \sum \langle \pi \rangle$ 

$[\Delta; 0]$	$\langle 0 \rangle, {}^\dagger \langle 0 \rangle$
$[\Delta; 1]$	$\langle 0 \rangle, {}^\dagger \langle 1 \rangle$
$[\Delta; 1^2]$	$\langle 0 \rangle, {}^\dagger (\langle 1^2 \rangle - \langle 0 \rangle)$
$[\Delta; 2]$	$\langle 0 \rangle, {}^\dagger (\langle 2 \rangle + \langle 1 \rangle + \langle 0 \rangle)$
$[\Delta; 1^3]$	$\langle 0 \rangle, {}^\dagger (\langle 1^3 \rangle - \langle 1 \rangle)$
$[\Delta; 21]$	$\langle 0 \rangle, {}^\dagger (\langle 21 \rangle + \langle 2 \rangle + \langle 1^2 \rangle)$
$[\Delta; 3]$	$\langle 0 \rangle, {}^\dagger (\langle 3 \rangle + \langle 2 \rangle + \langle 1^2 \rangle + 2\langle 1 \rangle + \langle 0 \rangle)$
$[\Delta; 1^4]$	$\langle 0 \rangle, {}^\dagger (\langle 1^4 \rangle - \langle 1^2 \rangle)$
$[\Delta; 21^2]$	$\langle 0 \rangle, {}^\dagger (\langle 21^2 \rangle + \langle 21 \rangle + \langle 1^3 \rangle - \langle 2 \rangle - \langle 1 \rangle)$
$[\Delta; 2^2]$	$\langle 0 \rangle, {}^\dagger (\langle 2^2 \rangle + \langle 21 \rangle + \langle 3 \rangle + 2\langle 2 \rangle - \langle 1^2 \rangle + \langle 1 \rangle + \langle 0 \rangle)$
$[\Delta; 31]$	$\langle 0 \rangle, {}^\dagger (\langle 31 \rangle + 2\langle 21 \rangle + \langle 2 \rangle + \langle 1^3 \rangle + \langle 2 \rangle + 3\langle 1^2 \rangle$ $\langle 1 \rangle - \langle 0 \rangle) .$
$[\Delta; 4]$	$\langle 0 \rangle, {}^\dagger (\langle 4 \rangle + \langle 21 \rangle + \langle 3 \rangle + 3\langle 2 \rangle + \langle 1^2 \rangle + 3\langle 1 \rangle + 2\langle 0 \rangle)$

#### 5.4 DIMENSIONS OF IRREPS OF $S_n$ IN REDUCED NOTATION

Having established the validity of the reduced notation for spin irreps of  $S_n$ , we next consider the  $n$ -independence of the dimensional formulas for the ordinary and spin irreps of  $S_n$  in terms reduced notation.

The formula for the dimension  $f^{[\lambda]}$  given in chapter 1 is

$$f^{[\lambda]} = n!/H(\lambda)$$

It is possible to display the  $n$ -independence of an irrep  $[\lambda]$  explicitly, taking advantage of the reduced notation to give (Butler and King, 1973)

$$f^{[\lambda]} = f_n^{<\mu>} = \frac{1}{H(\mu)} \prod_{i=1}^{p_\mu} (n - w_\mu - \mu_i + i) \quad (5.13)$$

where  $H(\mu)$  is now an  $n$ -independent function and the remaining product term is explicitly  $n$ -dependent.

For example

$$f^{[n-5, 2^2 1]} = f_n^{<2^2 1>} = \frac{1}{24} n(n-1)(n-3)(n-5)(n-6) \quad (5.14)$$

The factors in this expression correspond to the vanishing of the  $S$ -functions  $\{-5, 2^2 1\}$ ,  $\{-4, 2^2 1\}$ ,  $\{-2, 2^2 1\}$ ,  $\{0, 2^2 1\}$  and  $\{1, 2^2 1\}$ . Substituting  $n=2$  and  $n=4$  in (5.14) give  $f^{[1^2]} = f^{[1^4]} = 1$  by virtue of the identities  $\{-3, 2^2 1\} = -\{1^2\}$  and  $\{-1, 2^2 1\} = \{1^4\}$ . Setting  $n = 7, 8, \dots$ , yields  $f^{[2^3 1]} = 14$ ,  $f^{[3 2^2 1]} = 70 \dots$  etc.

The dimension formula for spin irreps of  $S_n$  for  $k$ -parts  $[\lambda_1, \lambda_2, \dots, \lambda_k]$ , also given in chapter 1, is

$$f^{[\lambda]'} = 2^{[(n-k)/2]} n! \prod_{i=1}^k (\lambda_i)^{-1} \prod_{1 \leq \ell < s \leq k} \left( \frac{\lambda_\ell - \lambda_s}{\lambda_\ell + \lambda_s} \right)$$

Taking advantage of the reduced notation, we find

$$f_n^{<\mu>'} = 2^{[(n-p_\mu-1)/2]} C_{w_\mu}^n \prod_i^{p_\mu} \left( \frac{n-w_\mu-\mu_i}{n-w_\mu+\mu_i} \right) f^{<\mu>'} \quad (5.15)$$

where  $C_{w_\mu}^n$  is a binomial coefficient and

$$f^{<\mu>'} = w_\mu! \prod_i^{p_\mu} (\mu_i!)^{-1} \prod_{1 \leq \ell < s \leq p_\mu} \left( \frac{\mu_\ell - \mu_s}{\mu_\ell + \mu_s} \right) \quad (5.16)$$

is an  $n$ -independent factor which we shall term the reduced dimension of the spin irreps of  $S_n$ . For one part  $<k>$ ,  $f^{<k>} = k!/k! = 1$ . The reduced dimensions for spin irrep of  $S_n$  for more than one part are given in table VIII.

The above results allow us to determine explicitly the  $n$ -dependence of the dimensional formula appropriate to any reduced spin irrep  $<\mu>'$ . Thus for  $<31>'$  we have  $f^{<31>'} = 2$  from table VIII and thence

$$f_n^{[n-4, 31]'} = f_n^{<31>'} = 2^{[(n-3)/2]} (n-2)(n-5)(n-7)/12 \quad (5.17)$$

The factors in this expression correspond to the vanishing of the  $Q$ -functions  $Q_{(331)}$  and  $Q_{(131)}$ . Substituting  $n=6$  in (5.17) gives  $f^{[321]}'$  by virtue of the identity  $Q_{(231)} = -Q_{(321)}$ . Setting  $n = 8, 9 \dots$  yields  $f^{[431]}' = 48$ ,

TABLE VIII: Reduced Dimensions for  $S_n$  Spin Irreps

$\langle \lambda \rangle$	$f^{\langle \lambda \rangle}$
21	1
31	2
41	3
32	2
51	4
42	5
321	2
61	5
52	9
43	5
421	7
71	6
62	14
53	14
521	16
431	12
81	7
72	20
63	28
54	14
621	30
531	42
432	12



$$f^{[531]'} = 336 \dots \text{etc.}$$

### 5.5 THE $S_n \downarrow S_{n-1}$ BRANCHING RULE

The  $S_n \downarrow S_{n-1}$  branching rule for ordinary irreps of  $S_n$  is well known and in the reduced notation amounts to

$$\langle \mu \rangle \downarrow \langle \mu \rangle + \langle \mu/1 \rangle \quad (5.18)$$

which is the  $n$ -independent statement of the branching rule.

The statement of a similar rule for the spin irreps of  $S_n$  is complicated by the existence of self-associated spin irreps  $[\lambda]'^{\dagger}$  ( $n-k$  even) and associated pairs of irreps  $[\lambda]_{\pm}'$  ( $n-k$  odd). Two special cases have been discussed (Wales 1979), but no general statement of the rule for spin irreps appears to have been given. The general rule follows by noting (5.18) and using (3.29) and theorem 1 of chapter 1 to give in the reduced notation

$$\langle \mu \rangle' \downarrow \langle \mu \rangle'^{\dagger} + \langle \mu/1 \rangle'^{\dagger} - \delta_{\mu_r, 1} \langle \mu_1, \mu_2, \dots, \mu_{r-1} \rangle \quad (5.19a)$$

$$\langle \mu \rangle'^{\dagger} \downarrow \langle \mu \rangle' + \langle \mu/1 \rangle' \quad (5.19b)$$

In (5.19a) the left-hand-term  $\langle \mu \rangle'$  has  $(n-p_{\mu}+1)$  odd while in (5.19b) the left-hand-term  $\langle \mu \rangle'^{\dagger}$  has  $(n-p_{\mu}+1)$  even.

The following two examples illustrate the application of (5.19). From (5.19a)

$$\langle 420 \rangle' \downarrow \langle 421 \rangle'^{\dagger} + \langle 321 \rangle'^{\dagger} + \langle 42 \rangle' \quad n \text{ odd}$$

and hence for

$$S_{13} \downarrow S_{12} [6421]' \downarrow [5421]'^{\dagger} + [6321]'^{\dagger} + [642]$$

From (5.19b)

$$\begin{aligned} \langle 421 \rangle'^{\dagger} \downarrow \langle 421 \rangle'^{\dagger} + \langle 42 \rangle'^{\dagger} + \langle 321 \rangle'^{\dagger} \\ = \langle 421 \rangle' + \langle 421 \rangle' + \langle 42 \rangle'^{\dagger} + \langle 321 \rangle' + \langle 321 \rangle' \\ n \text{ even} \end{aligned}$$

and hence for

$$\begin{aligned} S_{14} \downarrow S_{13} [7421]'^{\dagger} \downarrow [6421]' + \widetilde{[6421]}' + [742]'^{\dagger} \\ + [7321]' + \widetilde{[7321]}' \end{aligned}$$

## 5.6 KRONECKER PRODUCTS IN REDUCED NOTATION

### I. Kronecker products of ordinary irreps

Using branching rule  $O_n \downarrow S_n$ , Littlewood (1957b) has shown that the Kronecker product of symmetric group irrep  $\langle \lambda \rangle, \langle \mu \rangle$  in the reduced notation may be expressed in an  $N$ -independent form as

$$\langle \lambda \rangle \langle \mu \rangle = \sum_{\alpha, \beta, \gamma} \langle (\lambda/\alpha\beta) \cdot (\mu/\alpha\gamma) \cdot (\beta \circ \gamma) \rangle \quad (5.20)$$

where  $(\beta)$  and  $(\gamma)$  are necessarily partitions of the same weight. The inner product  $\beta \circ \gamma$  may be evaluated in terms of reduced Kronecker products  $\langle \sigma \rangle \langle \tau \rangle$  of weight

$w_\sigma w_\tau \ll w_\lambda w_\mu$  and hence (5.20) may be used to systematically build up reduced products using reduced products of lower weight.

Thus, for example,

$$\begin{aligned}
 \langle 1 \rangle \langle 1 \rangle &= \sum_{\alpha, \beta, \gamma} \langle (1/\alpha\beta) \cdot (1/\alpha\gamma) \cdot (\beta \circ \gamma) \rangle \\
 &= \langle (1/0) \cdot (1/0) \cdot (0 \circ 0) \rangle + \langle (1/1) \cdot (1/1) \cdot (1 \circ 1) \rangle \\
 &\quad + \langle (1/1) \cdot (1/1) \cdot (0 \circ 0) \rangle \\
 &= \langle 1 \cdot 1 \cdot 0 \rangle + \langle 0 \cdot 0 \cdot 1 \rangle + \langle 0 \cdot 0 \cdot 0 \rangle \\
 &= \langle 2 \rangle + \langle 1^2 \rangle + \langle 1 \rangle + \langle 0 \rangle
 \end{aligned}$$

similarly,

$$\begin{aligned}
 \langle 2 \rangle \langle 1^2 \rangle &= \sum_{\alpha\beta\gamma} \langle (2/\alpha\beta) \cdot (1^2/\alpha\gamma) \cdot (\beta \circ \gamma) \rangle \\
 &= \langle (2/0) \cdot (1^2/0) \cdot (0 \circ 0) \rangle \\
 &\quad + \langle (2/1) \cdot (1^2/1) \cdot (1 \circ 1) \rangle \\
 &\quad + \langle (2/1) \cdot (1^2/1) \cdot (0 \circ 0) \rangle \\
 &\quad + \langle (2/2) \cdot (1^2/1^2) \cdot (1 \circ 1) \rangle \\
 &\quad + \langle (2/2) \cdot (1^2/1^2) \cdot (2 \circ 1^2) \rangle \\
 &= \langle 2 \cdot 1^2 \rangle + \langle 1 \cdot 1 \cdot 1 \rangle + \langle 1 \cdot 1 \cdot 0 \rangle + \langle 0 \cdot 0 \cdot 1 \rangle + \langle 0 \cdot 0 \cdot 1^2 \rangle \\
 &= \langle 31 \rangle + \langle 21^2 \rangle + \langle 3 \rangle + 2\langle 21 \rangle + \langle 1^3 \rangle + \langle 2 \rangle + 2\langle 1^2 \rangle \\
 &\quad + \langle 1 \rangle
 \end{aligned}$$

allowing us to immediately write down for  $S_{30}$

$$\begin{aligned} [28,2][28,1^2] &= [26,31] + [26,21^2] + [27,3] \\ &+ 2[27,21] + [27,1^3] + [28,2] + 2[28,1^2] + [29,1] \end{aligned}$$

without any construction of the character table for  $S_{30}$  being required.

## II. Kronecker products of basic spin with ordinary irreps

In sec. 3.8 we defined the inner product of Q- and S-function in order to resolve the Kronecker products of basic spin with ordinary irreps for  $S_n$ .

In order to facilitate the reduced notation, we now introduce a raising operator  $R_{0j}$  which has the effect of decreasing  $\mu_j$  by one unit, we can then write a reduced version of (3.31) as

$$\langle \mu \rangle_q = \prod_{0 \leq j < j} (1 + R_{ij}) Q_{\langle \mu \rangle} \quad (5.21a)$$

$$Q_{\langle \mu \rangle} = \prod_{0 \leq j < j} (1 - R_{ij} + R_{ij}^2 + \dots) \langle \mu \rangle_q \quad (5.21b)$$

Equation (3.31) now becomes in reduced notation

$$Q_{\langle 0 \rangle} {}^o \langle \mu \rangle_h = \prod_{0 \leq i < j} (1 + R_{ij}) Q_{\langle \mu \rangle} \quad (5.22)$$

Application of (5.22) to an r-part reduced partition will yield  $e^{C_2^{r+1}}$  terms, not necessarily all distinct. The

results for  $r \leq 3$  are given in table IX. For a specific partition, non-standard Q-functions may arise and must be reduced to the standard descending order by use of the modification rules given earlier.

Thus in the case of a four-part reduced partition we expect (5.2) to yield 1024 terms. Specialisation to the reduced partition  $\langle 4321 \rangle$  results in the survival of just 88 Q-functions, and of these only 25 are distinct. Restrictions to a particular value of  $n$  may result in even fewer terms surviving. Thus if  $n = 15$  the product involving  $\langle 4321 \rangle$  yields a total of 56 Q-functions of which 15 are distinct.

We are now in a position to be able to give a complete algorithm to evaluate the Kronecker products  $\langle 0 \rangle'^{\dagger} \langle \mu \rangle$ , as follows.

#### Algorithm 1

- (1) Evaluate the S-function inner product

$$Q_{\langle 0 \rangle} \circ \langle \mu \rangle_h \text{ using (5.22)}$$

- (2) Standardize all non-standard Q-functions

- (3) Replace every Q-function,  $Q_{\langle 0 \rangle}$ , appearing in the product by

$$2^{[(K-n(\bmod 2))/2]} \langle \rho \rangle'^{\dagger}$$

If  $n - k$  is odd then  $\langle \rho \rangle'^{\dagger} \equiv \langle \rho \rangle' + \langle \tilde{\rho} \rangle'$

TABLE IX:  $Q_{<0>}^{\circ<\lambda>}_h$  inner products

$$Q_{<0>}^{\circ<p>}_h = Q_{<p>} + Q_{<p-1>}$$

$$Q_{<0>}^{\circ<pq>}_h = Q_{<pq>} + 2Q_{<pq-1>} + Q_{<pq-2>} + Q_{<p+1,q-1>} \\ + Q_{<p+1,q-2>} + Q_{<p-1q>} + Q_{<p-1q-1>}$$

$$Q_{<0>}^{\circ<pqr>}_h = Q_{<pqr>} + 4Q_{<pqr-1>} + 4Q_{<pqr-2>} + Q_{<pqr-3>} \\ + 2Q_{<pq-1r>} + 4Q_{<pq-1r-1>} + 2Q_{<pq-1r-2>} + Q_{<pq-2r>} \\ + Q_{<pq-2r-1>} + Q_{<pq+1r-1>} + 2Q_{<pq+1r-2>} + Q_{<pq+1r-3>} \\ + Q_{<p-1qr>} + 2Q_{<p-1qr-1>} + Q_{<p-1qr-2>} + Q_{<p-1q-1r>} \\ + Q_{<p-1q-1r-1>} + Q_{<p-1q+1r-1>} + Q_{<p-1q+1r-2>} \\ + 2Q_{<p+1qr-1>} + 4Q_{<p+1qr-2>} + 2Q_{<p+1qr-3>} \\ + Q_{<p+1q-1r>} + 4Q_{<p+1q-1r-1>} + 4Q_{<p+1q-1r-2>} \\ + Q_{<p+1q-1r-3>} + Q_{<p+1q-2r>} + 2Q_{<p+1q-2r-1>} \\ + Q_{<p+1q-2-2r-2>} + Q_{<p+1q+1r-2>} + Q_{<p+1q+1r-3>} \\ + Q_{<p+2qr-3>} + Q_{<p+2q-1r-1>} + 2Q_{<p+2q-1r-2>} \\ + Q_{<p+2q-1r-3>} + Q_{<p+2q-2r-1>} + Q_{<p+2q-2r-2>}$$

As an example we evaluate the  $\langle 0 \rangle^{\dagger} \langle 321 \rangle$ . First we note from table 3 that for  $\langle 321 \rangle$  we have

$$\begin{aligned} Q_{\langle 0 \rangle} \circ \langle 321 \rangle &= Q_{\langle 5 \rangle} + 2Q_{\langle 4 \rangle} + Q_{\langle 3 \rangle} + Q_{\langle 51 \rangle} + 2Q_{\langle 42 \rangle} \\ &+ 3Q_{\langle 32 \rangle} + 3Q_{\langle 41 \rangle} + 3Q_{\langle 31 \rangle} + Q_{\langle 21 \rangle} + Q_{\langle 321 \rangle} \end{aligned}$$

Specialisation to  $n = 11$  gives

$$\begin{aligned} [11]^{\dagger} [5321] &= [65]^{\dagger} + 2[74]^{\dagger} + [83]^{\dagger} + 4[542]^{\dagger} \\ &+ 6[632]^{\dagger} + 6[641]^{\dagger} + 6[731]^{\dagger} + 2[821]^{\dagger} \\ &+ 2[5321]^{\dagger} \end{aligned}$$

and may be checked by dimensions

$$\begin{aligned} 32 \times 2310 &= 1344 + 2 \times 2880 + 2400 + 4 \times 1760 \\ &+ 6 \times 2464 + 6 \times 3168 + 6 \times 3168 + 2 \times 1232 \\ &+ 2 \times 1056 = 73920 \end{aligned}$$

whereas for  $n = 10$  we find

$$\begin{aligned} [10]^{\dagger} [4321] &= 4[64]^{\dagger} + 2[73]^{\dagger} + 6[532]^{\dagger} + 4[541]^{\dagger} \\ &+ 6[631]^{\dagger} + 2[721]^{\dagger} + 4[4321]^{\dagger} \end{aligned}$$

and with dimension checking

$$\begin{aligned}
32 \times 768 &= 4 \times 672 + 2 \times 768 + 6 \times 864 + 4 \times 896 \\
&+ 6 \times 1600 + 2 \times 800 + 4 \times 96
\end{aligned}$$

We note that for  $n$  even  $\langle 0 \rangle'^{\dagger} = \langle 0 \rangle' + \langle \tilde{0} \rangle'$  and our algorithm will not of course yield the product  $\langle 0 \rangle' \langle \mu \rangle$ , except where  $\langle \mu \rangle$  corresponds to a self-associated irrep. To separate out the terms  $\langle 0 \rangle' \langle \mu \rangle$  and  $\langle \tilde{0} \rangle' \langle \mu \rangle$  from  $\langle 0 \rangle'^{\dagger} \langle \mu \rangle$  requires use of difference characters.

Our algorithm can be used to expand recursively any spin irrep  $\langle \mu \rangle'^{\dagger}$  as a linear combination of terms of the type  $\langle 0 \rangle'^{\dagger} \langle \lambda \rangle$ . For example, we may readily establish the results shown in Table X.

### III. Kronecker products of spin with ordinary irreps

We now consider the general Kronecker product of a spin irrep  $\langle \nu \rangle'^{\dagger}$  with an ordinary irrep  $\langle \mu \rangle$  with the understanding that  $\langle \nu \rangle'^{\dagger}$  is either a self-associated irrep ( $n-p_{\nu}-1$ ) even or a pair of associated irreps ( $n-p_{\nu}-1$ ) odd.

#### Algorithm 2

- (1) Expand  $Q_{\langle \nu \rangle}$  as a sum of S-functions using (5.21b)
- (2) Evaluate terms in the relevant S-function inner products using (5.20).
- (3) Express the resulting S-functions as Q-functions using (5.21a).
- (4) Standardize the Q-functions



TABLE X: Expansion of spin irreps<sup>†</sup> in terms of the basic spin irreps and ordinary irreps of  $S_n$ .

$$\langle 0 \rangle, {}^{\dagger} = \langle 0 \rangle, {}^{\dagger} \langle 0 \rangle$$

$$\langle 0 \rangle, {}^{\dagger} = \left(\frac{1}{2}\right) \langle 0 \rangle, {}^{\dagger} (\langle 1 \rangle - \langle 0 \rangle)$$

$$\langle 2 \rangle, {}^{\dagger} = \left(\frac{1}{2}\right) \langle 0 \rangle, {}^{\dagger} (\langle 2 \rangle - \langle 1 \rangle + \langle 0 \rangle)$$

$$\langle 21 \rangle, {}^{\dagger} = \frac{1}{2} \langle 0 \rangle, {}^{\dagger} (\langle 21 \rangle - \langle 3 \rangle - \langle 2 \rangle)$$

$$\langle 3 \rangle, {}^{\dagger} = \left(\frac{1}{2}\right) \langle 0 \rangle, {}^{\dagger} (\langle 3 \rangle - \langle 2 \rangle + \langle 1 \rangle - \langle 0 \rangle)$$

$$\langle 31 \rangle, {}^{\dagger} = \frac{1}{2} \langle 0 \rangle, {}^{\dagger} (\langle 31 \rangle - \langle 21 \rangle - \langle 4 \rangle + \langle 2 \rangle)$$

$$\langle 32 \rangle, {}^{\dagger} = \frac{1}{2} \langle 0 \rangle, {}^{\dagger} (\langle 32 \rangle - \langle 41 \rangle - \langle 31 \rangle + \langle 5 \rangle + \langle 4 \rangle + \langle 3 \rangle)$$

$$\langle 4 \rangle, {}^{\dagger} = \left(\frac{1}{2}\right) (\langle 4 \rangle - \langle 3 \rangle + \langle 2 \rangle - \langle 1 \rangle + \langle 0 \rangle)$$

$$\langle 41 \rangle, {}^{\dagger} = \frac{1}{2} \langle 0 \rangle, {}^{\dagger} (\langle 41 \rangle + \langle 31 \rangle + \langle 21 \rangle - \langle 5 \rangle - \langle 2 \rangle)$$

<sup>†</sup>where the coefficient appears as  $\left(\frac{1}{2}\right)$  it is to be included only for  $n$  even.

- (5) Replace each  $Q_{\langle \rho \rangle}$ , including  $Q_{\langle v \rangle}$  by  $2^{[(K-n(\bmod 2))/2]}_{\langle \rho \rangle}{}^{\dagger}$

If  $(n-p_{\rho}-1)$  is odd put  $\langle \rho \rangle{}^{\dagger} = \langle \rho \rangle + \langle \tilde{\rho} \rangle$ .

The implementation of the above algorithm is seen in the evaluation of  $\langle 2 \rangle{}^{\dagger} \langle 1^2 \rangle$  as follows

$$\begin{aligned} Q_{\langle 2 \rangle} \circ \langle 1^2 \rangle_h &= (\langle 2 \rangle_q - \langle 1 \rangle_q + \langle 0 \rangle_q) \circ \langle 1^2 \rangle \\ &+ \langle 21^2 \rangle_q + \langle 31 \rangle_q + \langle 21 \rangle_q + \langle 3 \rangle_q + 2\langle 1^2 \rangle_q \end{aligned}$$

Use of (5.21a) then gives

$$\langle 21^2 \rangle_q = Q_{\langle 31 \rangle} + Q_{\langle 21 \rangle} + Q_{\langle 4 \rangle} + 2Q_{\langle 3 \rangle} + 2Q_{\langle 2 \rangle} + Q_{\langle 1 \rangle}$$

$$\langle 31 \rangle_q = Q_{\langle 31 \rangle} + Q_{\langle 21 \rangle} + Q_{\langle 4 \rangle} + 2Q_{\langle 3 \rangle} + Q_{\langle 2 \rangle}$$

$$\langle 21 \rangle_q = Q_{\langle 21 \rangle} + Q_{\langle 3 \rangle} + 2Q_{\langle 2 \rangle} + Q_{\langle 1 \rangle}$$

$$\langle 3 \rangle_q = Q_{\langle 3 \rangle} + Q_{\langle 2 \rangle}$$

$$\langle 1^2 \rangle_q = Q_{\langle 2 \rangle} + Q_{\langle 1 \rangle} + Q_{\langle 0 \rangle}$$

and hence

$$\begin{aligned} Q_{\langle 2 \rangle} \circ \langle 1^2 \rangle_h &= 2Q_{\langle 31 \rangle} + 3Q_{\langle 21 \rangle} + 2Q_{\langle 4 \rangle} + 6Q_{\langle 3 \rangle} + 8Q_{\langle 2 \rangle} \\ &+ 4Q_{\langle 1 \rangle} + 2Q_{\langle 0 \rangle} \end{aligned}$$

and thus for n odd we have

$$\begin{aligned} \langle 2 \rangle^{\dagger} \langle 1^2 \rangle &= 2(2\langle 31 \rangle^{\dagger} + 3\langle 21 \rangle^{\dagger}) + 2\langle 4 \rangle^{\dagger} + 6\langle 3 \rangle^{\dagger} \\ &+ 8\langle 2 \rangle^{\dagger} + 4\langle 1 \rangle^{\dagger} + 2\langle 0 \rangle^{\dagger} \end{aligned}$$

and for n even

$$\begin{aligned} \langle 2 \rangle^{\dagger} \langle 1^2 \rangle &= 2\langle 31 \rangle^{\dagger} + 3\langle 21 \rangle^{\dagger} + 2\langle 4 \rangle^{\dagger} + 6\langle 3 \rangle^{\dagger} + 8\langle 2 \rangle^{\dagger} \\ &+ 4\langle 1 \rangle^{\dagger} + \langle 0 \rangle^{\dagger} \end{aligned}$$

The above algorithm successfully resolves any product  $\langle v \rangle^{\dagger} \langle \mu \rangle$ . It remains to consider the case where  $\langle v \rangle^{\dagger} = \langle v \rangle' + \langle \tilde{v} \rangle'$  which arises for  $(n-p_v-1)$  odd. Consideration of the properties of the difference characters introduced in chapter 1 leads to the following algorithm for evaluating the terms in  $\langle v \rangle' \langle \mu \rangle$  and  $\langle \tilde{v} \rangle' \langle \mu \rangle$ .

### Algorithm 3

- (1) Evaluate  $\langle v \rangle^{\dagger} \langle \mu \rangle$  using algorithm 2.
- (2) Divide the coefficients associated with every term found in (1) by 2. The integral part of the resulting coefficients is the number of times its corresponding irrep occurs in  $\langle v \rangle' \langle \mu \rangle$  and  $\langle \tilde{v} \rangle' \langle \mu \rangle$ . If there is no residue the resolution is complete.
- (3) The only possible residue will be a term  $\langle v \rangle^{\dagger} = \langle v \rangle' + \langle \tilde{v} \rangle'$ . If the characteristic  $\chi_{(v)}^{\langle \mu \rangle} = +1$ ,  $\langle v \rangle'$  is assigned to  $\langle v \rangle' \langle \mu \rangle$  while if  $\chi_{(v)}^{\langle \mu \rangle} = -1$  the opposite assignment is made.

The characteristic  $\chi_{(\nu)}^{<\mu>}$ , may be readily calculated by first noting that the class  $(n-w_{\nu}, (\nu))$  can only involve distinct cycles and

$$\chi_{(\nu)}^{<\mu>} = \chi_{(n-w_{\nu}, (\nu))}^{[n-w_{\mu}, (\mu)]} \quad (5.23)$$

The value of characteristic  $\chi_{(\rho)}^{[\pi]}$  may be found using the theorem due to Littlewood quoted in chapter 1.

As an example of the application of algorithm 3, we consider the evaluation of  $<2>'<1>$  and  $<\tilde{2}>'<1>$ . From use of algorithm 2 we deduce that for  $n$  odd

$$<2>'<1>^{\dagger} = 2<21>'^{\dagger} + 2<3>'^{\dagger} + 3<2>'^{\dagger} + 2<1>'^{\dagger}$$

while for  $n$  even

$$<2>'<1>^{\dagger} = 2<21>'^{\dagger} + 4<3>'^{\dagger} + 6<2>'^{\dagger} + 4<1>'^{\dagger}$$

For  $n$  even the above resolution is complete since  $<2>'^{\dagger}$  is self-associated. For  $n$  odd we must have

$$<2>'<1> \supset <21>'^{\dagger} + <3>'^{\dagger} + <2>'^{\dagger} + <1>'^{\dagger}$$

and

$$<\tilde{2}>'<1> \supset <21>'^{\dagger} + <3>'^{\dagger} + <2>'^{\dagger} + <1>'^{\dagger}$$

The residue is  $<2>'^{\dagger} = <2>' + <\tilde{2}>'$ . We need to evaluate  $\chi_{(2)}^{<1>}$ . Consider  $\chi_{(32)}^{[41]}$ ; use of the littlewood's theorem yields the diagram

1	2	2	2
1			

with just one vertical step and hence

$$\chi_{(2)}^{<1>} = -1$$

leading, for  $n$  odd to

$$<2>'<1> = <21>'^\dagger + <3>'^\dagger + <2>'^\dagger + <1>'^\dagger + <\tilde{2}>'$$

$$<\tilde{2}>'<1> = <21>'^\dagger + <3>'^\dagger + <2>'^\dagger + <1>'^\dagger + <2>'$$

#### IV. Kronecker Products of spin irreps

It remains now to develop an algorithm for resolving the Kronecker product of a spin irrep  $<\mu>'$  with another spin irrep  $<\nu>'$  into a sum of ordinary irreps. To this end we first compute  $<\mu>'^\dagger <\nu>'^\dagger$  using the following algorithm.

##### Algorithm 4

(1) Expand  $<\mu>'^\dagger$  and  $<\nu>'^\dagger$  as products of the basic spin irreps  $<0>'^\dagger$  with ordinary irreps to yield

$$<\mu>'^\dagger = <0>'^\dagger (g_\mu^\pi <\pi>)$$

$$<\nu>'^\dagger = <0>'^\dagger (g_\nu^\sigma <\sigma>)$$

(2) The product  $\langle 0 \rangle^{\dagger 2}$  for  $S_{2v+1}$  is evaluated as

$$\langle 0 \rangle^{\dagger 2} = \sum_{x=0}^v \langle 1^x \rangle^{\dagger} \quad (5.24a)$$

and for  $S_{2v}$  as

$$\langle 0 \rangle^{\dagger 2} = 2 \sum_{x=0}^{v=1} \langle 1^x \rangle^{\dagger} \quad (5.24b)$$

(3) The calculation is now reduced to the evaluation of Kronecker products of ordinary irreps and may be effected by use of (5.20).

Use of the above algorithm readily leads to the n-independent results

$$\langle 1 \rangle^{\dagger} \langle 1 \rangle^{\dagger} = \langle 0 \rangle^{\dagger 2} (\langle 2 \rangle + \langle 1^2 \rangle + 2\langle 0 \rangle - \langle 1 \rangle)$$

and

$$\langle 2 \rangle^{\dagger} \langle 1 \rangle^{\dagger} = \langle 0 \rangle^{\dagger 2} (\langle 3 \rangle + \langle 21 \rangle + 2\langle 1 \rangle - \langle 2 \rangle - 2\langle 0 \rangle)$$

with the right-hand-side being divided by four for n even.

If the above results are specialized to  $S_7$  we find

$$\begin{aligned} [61]^{\dagger} [61]^{\dagger} &= 2[7]^{\dagger} + 4[61]^{\dagger} + 6[52]^{\dagger} + 8[51^2]^{\dagger} + 4[43]^{\dagger} \\ &\quad + 10[421]^{\dagger} + 8[41^3]^{\dagger} + 4[32^2]^{\dagger} \end{aligned}$$

and

$$\begin{aligned} [62]^{\dagger} [61]^{\dagger} &= 4[61]^{\dagger} + 8[52]^{\dagger} + 8[51^2]^{\dagger} + 8[43]^{\dagger} \\ &\quad + 20[421]^{\dagger} + 12[41^3]^{\dagger} + 12[32^2]^{\dagger} \end{aligned}$$

where we understand that if  $[\lambda] \equiv [\tilde{\lambda}]$  then  $[\lambda]^{\dagger} = [\lambda]$  otherwise  $[\lambda]^{\dagger} = [\lambda] + [\tilde{\lambda}]$ .

The  $[61]'$  and  $[52]'$  irreps of  $S_7$  constitute pairs of associated irreps. To resolve the products it is necessary to consider the properties of the difference characters.

Consider the product  $\langle \mu \rangle' \langle \nu \rangle'$  where  $\mu \neq \nu$  and the irreps are associated irreps. Since

$$\langle \mu \rangle' \langle \nu \rangle'' = \langle \mu \rangle'' \langle \nu \rangle' = 0 \quad (5.25a)$$

and

$$\langle \tilde{\mu} \rangle' \langle \nu \rangle'' = \langle \mu \rangle'' \langle \tilde{\nu} \rangle' = 0 \quad (5.25b)$$

we have

$$\langle \mu \rangle' \langle \nu \rangle' = \langle \mu \rangle' \langle \tilde{\nu} \rangle' = \frac{1}{4} (\langle \mu \rangle'^{\dagger} \langle \nu \rangle'^{\dagger}) \quad (5.26)$$

and hence if  $\mu \neq \nu$  we may trivially resolve the Kronecker products. Thus

$$\begin{aligned} [52]'[61]' &= [52]' \widetilde{[61]}' \\ &= [61]^{\dagger} + 2[52]^{\dagger} + 2[51^2]^{\dagger} + 2[43]^{\dagger} + 5[521]^{\dagger} \\ &\quad + 3[41^3]^{\dagger} + 3[32^2]^{\dagger} \end{aligned}$$

when  $\mu = \nu$  we have

$$\langle \mu \rangle' \langle \mu \rangle' = \frac{1}{4} [\langle \mu \rangle'^{\dagger 2} + \langle \mu \rangle''^2] \quad (5.27a)$$

$$\begin{aligned} \langle \mu \rangle' \langle \tilde{\mu} \rangle' &= \frac{1}{4} [\langle \mu \rangle'^{\dagger 2} - \langle \mu \rangle''^2] \\ &= \langle \tilde{0} \rangle' (\langle \mu \rangle'^2) \end{aligned} \quad (5.27b)$$

The problem is solved once  $\langle \mu \rangle'^2$  is resolved. Consider the character of  $\langle \mu \rangle'^2$ . This will have non-zero character only for the class  $(\lambda_1, \dots, \lambda_k)$ . Let  $\langle \mu \rangle'^2 = g_{\mu\mu}^{\rho} \langle \rho \rangle$ ; in each of this class

$$(\chi_{(\mu)}^{\langle \mu \rangle'})^2 = i^{n-k+1} \lambda_1 \lambda_2 \dots \lambda_k = g_{\mu\mu}^{\rho} \chi_{(\mu)}^{\langle \rho \rangle} \quad (5.28)$$

and  $n-k+1$  is even, so

$$g_{\mu\mu}^{\rho} = 2i^{n-k+1} \chi_{(\lambda)}^{[n-w]_{\rho}, \rho}$$

The character  $\chi_{(\lambda)}^{[n-w]_{\rho}, \rho}$  may be found from Littlewood's theorem.

In the case of  $S_7$  we easily find

$$[61]'' = 2([1^7] - [7] + [52] - [2^2 1^3] - [421] + [321^2])$$

and hence

$$\begin{aligned} [61]'[61]' &= [61]^{\dagger} + [52]^{\dagger} + [52] + 2[51^2]^{\dagger} + 2[41^3]^{\dagger} \\ &\quad + [43]^{\dagger} + 2[421]^{\dagger} + [421] + [32^2]^{\dagger} + [\tilde{7}] \end{aligned}$$

$$\begin{aligned} [61]'[61]' &= [61]^{\dagger} + [52]^{\dagger} + \widetilde{[52]} + 2[51^2]^{\dagger} + 2[41^3]^{\dagger} \\ &\quad + [43]^{\dagger} + 2[421]^{\dagger} + \widetilde{[421]} + [32^2]^{\dagger} + [7] \end{aligned}$$

As a consequence of the preceding, it is evident that any Kronecker product involving spin irreps may be systematically resolved. Explicit determination of the difference characters is only required for the special case of  $\mu = \nu$ .



### 5.7 SYMMETRIZED KRONECKER SQUARES AND THE CLASSIFICATION OF IRREPS OF $S_n$

The Kronecker square of an ordinary irrep

$$[\lambda] = [n-w_\mu, (\mu)] = \langle \mu \rangle \quad (5.29)$$

of  $S_n$  may be resolved into its symmetric parts

$$\langle \mu \rangle \otimes \{2\} = \{\lambda\} \otimes \{2\} \quad (5.30a)$$

and its antisymmetric parts

$$\langle \mu \rangle \otimes \{1^2\} = \{\lambda\} \otimes \{1^2\} \quad (5.30b)$$

Using Littlewood's result (Littlewood 1958b, Butler and King 1973)

$$\begin{aligned} \{\lambda\} \otimes \{\tau\} &= \langle \mu \rangle \otimes \{\tau\} \\ &= \sum_{\alpha} \sum_{\beta < \gamma} \langle (\mu/\alpha\beta) \cdot (\mu/\alpha\gamma) \cdot (\beta \circ \gamma) \rangle \\ &\quad + \sum_{\alpha\beta\sigma} \langle (\mu/\alpha\beta) \otimes \{\sigma\} \cdot \{\beta\} \otimes (\sigma \circ \tau) \rangle \end{aligned} \quad (5.31)$$

For Kronecker square the  $\{\tau\}$  is  $\{2\}$  and  $\{1^2\}$  and the summation over only includes  $\{\sigma\} = \{2\}$  and  $\{1^2\}$ .

By way of an example we have

$$\begin{aligned}
\langle 2 \rangle \otimes \{2\} &= \sum_{\alpha} \sum_{\beta < \gamma} \langle 2/\alpha\beta \rangle \cdot \langle 2/\alpha\gamma \rangle \cdot \langle \beta \circ \gamma \rangle \\
&+ \sum_{\alpha\beta\sigma} \langle (2/\alpha\beta) \otimes \{\sigma\} \cdot \{\beta\} \otimes (\sigma \circ 2) \rangle
\end{aligned}$$

The first term is null since  $\beta < \gamma$  is first satisfied by  $\beta = (1^2)$  and  $\gamma = (2)$  but  $(2/1^2) = 1$ . In the second term  $(\sigma \circ 2)$  can only assume the values  $(2 \circ 2) = (2)$  and  $(1^2 \circ 2) = (1^2)$ . Noting that  $\langle 0 \rangle \otimes \{2\} = \langle 0 \rangle$  while  $\langle 0 \rangle \otimes \{1^2\} = 0$  leaves the non null terms as just

$$\begin{aligned}
\langle 2 \rangle \otimes \{2\} &= \langle (4) + (2^2) \rangle + \langle 2 \rangle + \langle 2.1 \rangle + \langle 0.2 \rangle \\
&+ \langle 0.1 \rangle + \langle 0.0 \rangle \\
&= \langle 4 \rangle + \langle 2^2 \rangle + \langle 3 \rangle + \langle 21 \rangle + 2\langle 2 \rangle + \langle 1 \rangle + \langle 0 \rangle
\end{aligned}$$

Likewise

$$\langle 2 \rangle \otimes \{1^2\} = \langle 31 \rangle + \langle 21 \rangle + \langle 1^3 \rangle + \langle 1^2 \rangle$$

In table XI, we give the resolution of Kronecker squares into their symmetric and antisymmetric parts for  $0 < \omega_{\mu} \leq 3$  (Butler and King 1973). Thus in  $S_8$  we have

$$\begin{aligned}
\{62\} \otimes \{2\} &= [4^2] + [42^2] + [53] + [521] + 2[62] + [71] \\
&+ [8]
\end{aligned}$$

and

$$\{62\} \otimes \{1^2\} = [431] + [521] + [51^3] + [61^2]$$

It is of interest to determine in which part of the resolution of a Kronecker square the identity irrep  $\langle 0 \rangle$

TABLE XI: Symmetrized Kronecker squares for the ordinary  
irreps of  $S_n$

$$\langle 0 \rangle \otimes \{2\} = \langle 0 \rangle$$

$$\langle 0 \rangle \otimes \{1^2\} = 0$$

$$\langle 1 \rangle \otimes \{2\} = \langle 0 \rangle + \langle 1 \rangle + \langle 2 \rangle$$

$$\langle 1 \rangle \otimes \{1^2\} = \langle 1^2 \rangle$$

$$\langle 2 \rangle \otimes \{2\} = \langle 0 \rangle + \langle 1 \rangle + 2\langle 2 \rangle + \langle 21 \rangle + \langle 3 \rangle + \langle 2^2 \rangle + \langle 4 \rangle$$

$$\langle 2 \rangle \otimes \{1^2\} = \langle 1^2 \rangle + \langle 1^3 \rangle + \langle 21 \rangle + \langle 31 \rangle$$

$$\langle 1^2 \rangle \otimes \{2\} = \langle 0 \rangle + \langle 1 \rangle + 2\langle 2 \rangle + \langle 21 \rangle + \langle 3 \rangle + \langle 1^4 \rangle + \langle 2^2 \rangle$$

$$\langle 1^2 \rangle \otimes \{1^2\} = \langle 1^2 \rangle + \langle 1^3 \rangle + \langle 21 \rangle + \langle 21^2 \rangle$$

$$\begin{aligned} \langle 3 \rangle \otimes \{2\} &= \langle 0 \rangle + \langle 1 \rangle + 2\langle 2 \rangle + \langle 21 \rangle + 2\langle 3 \rangle + 2\langle 2^2 \rangle \\ &\quad + \langle 31 \rangle + 2\langle 4 \rangle + \langle 2^2 1 \rangle + \langle 32 \rangle + \langle 41 \rangle \\ &\quad + \langle 5 \rangle + \langle 2^3 \rangle + \langle 42 \rangle + \langle 6 \rangle \end{aligned}$$

$$\begin{aligned} \langle 3 \rangle \otimes \{1^2\} &= \langle 1^2 \rangle + \langle 1^3 \rangle + \langle 21 \rangle + \langle 21^2 \rangle + 2\langle 31 \rangle + \langle 31^2 \rangle \\ &\quad + \langle 32 \rangle + \langle 41 \rangle + \langle 3^2 \rangle + \langle 51 \rangle \end{aligned}$$

$$\begin{aligned} \langle 21 \rangle \otimes \{2\} &= \langle 0 \rangle + 2\langle 1 \rangle + \langle 1^2 \rangle + 4\langle 2 \rangle + \langle 1^3 \rangle + 5\langle 21 \rangle \\ &\quad + 4\langle 3 \rangle + 2\langle 1^4 \rangle + 3\langle 21^2 \rangle + 5\langle 2^2 \rangle \\ &\quad + 4\langle 31 \rangle + 3\langle 4 \rangle + \langle 1^5 \rangle + 2\langle 21^3 \rangle + 3\langle 2^2 1 \rangle \\ &\quad + 2\langle 31^2 \rangle + 3\langle 32 \rangle + 2\langle 41 \rangle + \langle 5 \rangle + \langle 2^3 \rangle \\ &\quad + \langle 31^3 \rangle + \langle 321 \rangle + \langle 42 \rangle \end{aligned}$$

$$\begin{aligned} \langle 21 \rangle \otimes \{1^2\} &= 3\langle 1^2 \rangle + 4\langle 1^3 \rangle + 4\langle 21 \rangle + \langle 3 \rangle + \langle 1^4 \rangle \\ &\quad + 6\langle 21^2 \rangle + \langle 2^2 \rangle + 5\langle 31 \rangle + 2\langle 21^3 \rangle \\ &\quad + 2\langle 2^2 1 \rangle + 4\langle 31^2 \rangle + 2\langle 32 \rangle + 2\langle 41 \rangle \\ &\quad + \langle 2^2 1^2 \rangle + \langle 321 \rangle + \langle 3^2 \rangle + \langle 41^2 \rangle \end{aligned}$$

$$\begin{aligned}
\langle 1^3 \rangle \otimes \{2\} &= \langle 0 \rangle + \langle 1 \rangle + 2\langle 2 \rangle + \langle 21 \rangle + 2\langle 3 \rangle + \langle 1^4 \rangle \\
&\quad + 2\langle 2^2 \rangle + \langle 31 \rangle + \langle 4 \rangle + \langle 1^5 \rangle + \langle 21^3 \rangle + \langle 2^2 1 \rangle \\
&\quad + \langle 32 \rangle + \langle 21^4 \rangle + \langle 2^3 \rangle
\end{aligned}$$

$$\begin{aligned}
\langle 1^3 \rangle \otimes \{1^2\} &= \langle 1^2 \rangle + \langle 1^3 \rangle + \langle 21 \rangle + 2\langle 21^2 \rangle + \langle 31 \rangle \\
&\quad + \langle 21^3 \rangle + \langle 2^2 1 \rangle + \langle 31^2 \rangle + \langle 1^6 \rangle + \langle 2^2 1^2 \rangle
\end{aligned}$$

of  $S_n$  occurs. Using (5.31) it is clear that  $\langle \lambda \rangle \otimes \{\tau\}$  with  $\{\tau\} = \{2\}$  or  $\{1^2\}$  will contain  $\langle 0 \rangle$  if and only if in the second summation we have  $\{\beta\} = 0$ ,  $\{\alpha\} = \{\lambda\}$  and the corresponding term  $\langle \{0\} \otimes \{\sigma\} \cdot \{0\} \otimes \{\sigma \circ \tau\} \rangle$  contains  $\langle 0 \rangle$ . This requires  $\{\sigma\} = \{\tau\} = \{2\}$ , so that

$$\langle \lambda \rangle \otimes \{2\} \supset \langle 0 \rangle$$

while

$$\langle \lambda \rangle \otimes \{1^2\} \not\supset \langle 0 \rangle$$

This means that every ordinary irrep of  $S_n$  is orthogonal.

We now consider the problem of resolving the Kronecker square of the spin irreps of  $S_n$  into its symmetric and antisymmetric terms. Recalling that  $[1^r] = [1^{n-r}]^\dagger$ , the formulas (4.33a,b) and (4.34a,b) for  $O_{2v+1}$  may be rewritten as

$$\Delta \otimes \{2\} = \sum_{\lambda} [1^{2v-4x}] \quad (5.32a)$$

$$\Delta \otimes \{1^2\} = \sum_x [1^{2v-2-4x}] \quad (5.32b)$$

for  $v \equiv 0, 1 \pmod{4}$ , while

$$\Delta \otimes \{2\} = \sum_x [1^{2v-2-4x}] \quad (5.33a)$$

$$\Delta \otimes \{1^2\} = \sum_x [1^{2v-4x}] \quad (5.33b)$$

for  $v \equiv 2, 3 \pmod{4}$ .

Consideration of the branching rules

$$O_{2\nu+1} \downarrow S_{2\nu+1}$$

$$[1^r] \downarrow \langle 1^r \rangle + \langle 1^{r-1} \rangle$$

and

$$\Delta \downarrow \langle 0 \rangle, {}^\dagger$$

then leads to

$$\langle 0 \rangle, {}^\dagger \otimes \{2\} = \sum_x (\langle 1^{2\nu-4x} \rangle + \langle 1^{2\nu-1-4x} \rangle) \quad (5.34a)$$

$$\langle 0 \rangle, {}^\dagger \otimes \{1^2\} = \sum_x (\langle 1^{2\nu-2-4x} \rangle + \langle 1^{2\nu-3-4x} \rangle) \quad (5.34b)$$

for  $\nu \equiv 0, 1 \pmod{4}$  and

$$\langle 0 \rangle, {}^\dagger \otimes \{2\} = \sum_x (\langle 1^{2\nu-2-4x} \rangle + \langle 1^{2\nu-1-4x} \rangle) \quad (5.35a)$$

$$\langle 0 \rangle, {}^\dagger \otimes \{1^2\} = \sum_x (\langle 1^{2\nu-4x} \rangle + \langle 1^{2\nu-1-4x} \rangle) \quad (5.35b)$$

for  $\nu \equiv 2, 3 \pmod{4}$ .

Inspection of (5.34) and (5.35) shows that for  $S_{2\nu+1}$  the basic spin irrep  $\langle 0 \rangle, {}^\dagger$  is orthogonal if  $\nu \equiv 0, 3 \pmod{4}$  while it is symplectic if  $\nu \equiv 1, 2 \pmod{4}$ .

Here we give some examples.

For  $S_5$ ,  $\nu = 2$  and  $\nu \equiv 2 \pmod{4}$  then

$$\langle 0 \rangle, {}^\dagger \otimes \{2\} = \langle 1^2 \rangle + \langle 1 \rangle$$

$$\langle 0 \rangle, {}^\dagger \otimes \{1^2\} = \langle 1^4 \rangle + \langle 1^3 \rangle + \langle 0 \rangle$$

So we have

$$[5]^{,\dagger} \otimes \{2\} = [31^2] + [41]$$

$$[5]^{,\dagger} \otimes \{1^2\} = [1^5] + [21^3] + [5]$$

$$[5] \subset [5]^{,\dagger} \otimes \{1^2\}$$

$$[5]^{,\dagger} \text{ is symplectic}$$

For  $S_7$ ,  $v = 3$  and  $v \equiv 3 \pmod{4}$

$$\langle 0 \rangle^{,\dagger} \otimes \{2\} = \langle 1^4 \rangle + \langle 1^3 \rangle + \langle 0 \rangle$$

$$\langle 0 \rangle^{,\dagger} \otimes \{1^2\} = \langle 1^6 \rangle + \langle 1^5 \rangle + \langle 1^2 \rangle + \langle 1 \rangle$$

So we have

$$[7]^{,\dagger} \otimes \{2\} = [31^4] + [41^3] + [7]$$

$$[7]^{,\dagger} \otimes \{1^2\} = [1^7] + [21^5] + [51^2] + [61]$$

$$[7] \subset [7]^{,\dagger} \otimes \{2\}$$

$$[7]^{,\dagger} \text{ is orthogonal}$$

For  $S_9$ ,  $v = 4$  and  $v \equiv 0 \pmod{4}$

$$\langle 0 \rangle^{,\dagger} \otimes \{2\} = \langle 1^8 \rangle + \langle 1^7 \rangle + \langle 1^4 \rangle + \langle 1^3 \rangle + \langle 0 \rangle$$

$$\langle 0 \rangle^{,\dagger} \otimes \{1^2\} = \langle 1^6 \rangle + \langle 1^5 \rangle + \langle 1^2 \rangle + \langle 1 \rangle$$

So we have

$$[9]^{,\dagger} \otimes \{2\} = [1^9] + [21^7] + [51^4] + [61^3] + [9]$$

$$[9]^{,\dagger} \otimes \{1^2\} = [31^6] + [41^5] + [71^2] + [81]$$

$$[9] \subset [9]^{,\dagger} \otimes \{2\}$$

$$[9]^{,\dagger} \text{ is orthogonal}$$

For  $S_{11}$ ,  $v = 5$  and  $v \equiv 1 \pmod{4}$

$$\langle 0 \rangle^{,\dagger} \otimes \{2\} = \langle 1^{10} \rangle + \langle 1^9 \rangle + \langle 1^6 \rangle + \langle 1^5 \rangle + \langle 1^2 \rangle + \langle 1 \rangle$$

$$\langle 0 \rangle^{,\dagger} \otimes \{1^2\} = \langle 1^8 \rangle + \langle 1^7 \rangle + \langle 1^4 \rangle + \langle 1^3 \rangle + \langle 0 \rangle$$

So we have

$$[11]^{,\dagger} \otimes \{2\} = [1^{11}] + [21^9] + [51^6] + [61^5] + [91^2] + [10,1]$$

$$[11]^{,\dagger} \otimes \{1^2\} = [31^8] + [41^7] + [71^4] + [81^3] + [11]$$

$$[11] \subset [11]^{,\dagger} \otimes \{1^2\}$$

$$[11]^{,\dagger} \text{ is symplectic}$$

In exactly the same way we find for  $O_{2v}$

$$\Delta \otimes \{2\} = \sum_x ([1^{2v-4x}] + [1^{2v-3-4x}]) \quad (5.36a)$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{2v-1-4x}] + [1^{2v-2-4x}]) \quad (5.36b)$$

for  $v \equiv 0 \pmod{4}$ , while

$$\Delta \otimes \{2\} = \sum_x ([1^{2v-4x}] + [1^{2v-1-4x}]) \quad (5.37a)$$



$$\Delta \otimes \{1^2\} = \sum_x ([1^{2\nu-2-4x}] + [1^{2\nu-3-4x}]) \quad (5.37b)$$

for  $\nu \equiv 1 \pmod{4}$ , and

$$\Delta \otimes \{2\} = \sum_x ([1^{2\nu-1-4x}] + [1^{2\nu-2-4x}]) \quad (5.38a)$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{2\nu-4x}] + [1^{2\nu-3-4x}]) \quad (5.38b)$$

for  $\nu \equiv 2 \pmod{4}$ , and

$$\Delta \otimes \{2\} = \sum_x ([1^{2\nu-2-4x}] + [1^{2\nu-3-4x}]) \quad (5.39a)$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{2\nu-4x}] + [1^{2\nu-1-4x}]) \quad (5.39b)$$

for  $\nu \equiv 3 \pmod{4}$ .

The branching rules for  $O_{2\nu} \downarrow S_{2\nu}$  then lead to the results (writing  $\langle 0 \rangle'^\dagger = \langle 0 \rangle' + \langle \tilde{0} \rangle'$ ).

$$\begin{aligned} \langle 0 \rangle'^\dagger \otimes \{2\} &= \sum_x (\langle 1^{2\nu-4x} \rangle + \langle 1^{2\nu-1-4x} \rangle + \langle 1^{2\nu-3-4x} \rangle \\ &\quad + \langle 1^{2\nu-4-4x} \rangle) \end{aligned} \quad (5.40a)$$

$$\langle 0 \rangle'^\dagger \otimes \{1^2\} = \sum_x (\langle 1^{2\nu-1-4x} \rangle + 2\langle 1^{2\nu-2-4x} \rangle + \langle 1^{2\nu-3-4x} \rangle) \quad (5.40b)$$

for  $\nu \equiv 0 \pmod{4}$ , while

$$\langle 0 \rangle'^\dagger \otimes \{2\} = \sum_x (\langle 1^{2\nu-4x} \rangle + 2\langle 1^{2\nu-1-4x} \rangle + \langle 1^{2\nu-2-4x} \rangle) \quad (5.41a)$$

$$\langle 0 \rangle'^\dagger \otimes \{1^2\} = \sum_x (\langle 1^{2\nu-2-4x} \rangle + 2\langle 1^{2\nu-3-4x} \rangle + \langle 1^{2\nu-4-4x} \rangle) \quad (5.41b)$$

for  $v \equiv 1 \pmod{4}$ , and

$$\langle 0 \rangle^{\dagger} \otimes \{2\} = \sum_x (\langle 1^{2v-1-4x} \rangle + 2\langle 1^{2v-2-4x} \rangle + \langle 1^{2v-3-4x} \rangle) \quad (5.42a)$$

$$\begin{aligned} \langle 0 \rangle^{\dagger} \otimes \{1^2\} = \sum_x (\langle 1^{2v-4x} \rangle + \langle 1^{2v-1-4x} \rangle + \langle 1^{2v-3-4x} \rangle \\ + \langle 1^{2v-4-4x} \rangle) \end{aligned} \quad (5.42b)$$

for  $v \equiv 2 \pmod{4}$ , and

$$\langle 0 \rangle^{\dagger} \otimes \{2\} = \sum_x (\langle 1^{2v-2-4x} \rangle + 2\langle 1^{2v-3-4x} \rangle + \langle 1^{2v-4-4x} \rangle) \quad (5.43a)$$

$$\langle 0 \rangle^{\dagger} \otimes \{1^2\} = \sum_x (\langle 1^{2v-4x} \rangle + 2\langle 1^{2v-1-4x} \rangle + \langle 1^{2v-2-4x} \rangle) \quad (5.43b)$$

for  $v \equiv 3 \pmod{4}$ .

In the case of  $S_{2v}$  the basic spin irrep is an associated pair of irreps  $\langle 0 \rangle^{\dagger} = \langle 0 \rangle' + \langle \tilde{0} \rangle'$  and it is necessary to use difference characters to complete the analysis of the Kronecker square. We have

$$\langle 0 \rangle'^2 = \frac{1}{4}(\langle 0 \rangle^{\dagger 2} + \langle 0 \rangle''^2) \quad (5.44)$$

where

$$\langle 0 \rangle^{\dagger 2} = 2 \sum_{x=0}^{2v-1} \langle 1^x \rangle$$

and

$$\langle 0 \rangle''^2 = 2i \sum_{x=0}^{2v-1} (-1)^x \langle 1^x \rangle$$

The square of the difference character  $\langle 0 \rangle''^2$  may be analyzed into its symmetric and antisymmetric parts to give

$$\begin{aligned}
\langle 0 \rangle' " \otimes \{2\} &= (\langle 0 \rangle' - \langle \tilde{0} \rangle') \otimes \{2\} \\
&= \langle 0 \rangle' \otimes \{2\} + \langle \tilde{0} \rangle' \otimes \{1^2\} - \langle 0 \rangle' \langle \tilde{0} \rangle'
\end{aligned}$$

where  $\langle \tilde{0} \rangle' = \epsilon \langle 0 \rangle'$  and  $\epsilon$  is the alternating irrep and

$$\begin{aligned}
\langle \tilde{0} \rangle' \otimes \{1^2\} &= (\epsilon \langle 0 \rangle') \otimes \{1^2\} = (\epsilon \otimes \{2\}) (\langle 0 \rangle' \otimes \{1^2\}) \\
&= \langle 0 \rangle' \otimes \{1^2\}
\end{aligned}$$

So we have

$$\langle 0 \rangle' " \otimes \{2\} = \langle 0 \rangle'^2 - \langle 0 \rangle' \langle \tilde{0} \rangle'$$

Similarly we have

$$\langle 0 \rangle' " \otimes \{1^2\} = \langle 0 \rangle'^2 - \langle 0 \rangle' \langle \tilde{0} \rangle'$$

and hence

$$\langle 0 \rangle' " \otimes \{2\} = \langle 0 \rangle' " \otimes \{1^2\} = \langle 0 \rangle'^2 / 2 \quad (5.45)$$

The Kronecker square of the basic spin irreps  $\langle 0 \rangle'$  of  $S_{2v}$  may be evaluated using (5.44). The resolution of the Kronecker square into its symmetric and antisymmetric terms then follows by noting that

$$\begin{aligned}
\langle 0 \rangle' \otimes \{2\} &= \frac{1}{2} (\langle 0 \rangle'^{\dagger} \otimes \{2\} + \langle 0 \rangle' " \otimes \{2\} - \langle 0 \rangle'^2) \\
&= \frac{1}{2} (\langle 0 \rangle'^{\dagger} \otimes \{2\} + \langle 0 \rangle'^2 / 2 - \langle 0 \rangle'^2) \quad (5.46a)
\end{aligned}$$

$$\langle 0 \rangle' \otimes \{1^2\} = \frac{1}{2}(\langle 0 \rangle'^{\dagger} \otimes \{1^2\} + \langle 0 \rangle'^{''2}/2 - \langle 0 \rangle'^2) \quad (5.46b)$$

and allows us to conclude that

$$\text{if } \nu \equiv 1, 3 \pmod{4} \quad \langle 0 \rangle' \text{ and } \langle \tilde{0} \rangle' \text{ are complex} \quad (5.47a)$$

while

$$\text{if } \nu \equiv 0 \pmod{4} \quad \langle 0 \rangle' \text{ and } \langle \tilde{0} \rangle' \text{ are orthogonal} \quad (5.47b)$$

or

$$\nu \equiv 2 \pmod{4} \quad \langle 0 \rangle' \text{ and } \langle \tilde{0} \rangle' \text{ are symplectic} \quad (5.47c)$$

Following examples are given for  $S_{2\nu}$ .

For  $S_4$ ,  $\nu = 2$  and  $\nu \equiv 2 \pmod{4}$  according to (5.42)

we have

$$\langle 0 \rangle'^{\dagger} \otimes \{2\} = \langle 1^3 \rangle + 2\langle 1^2 \rangle + \langle 1 \rangle$$

$$\langle 0 \rangle'^{\dagger} \otimes \{2\} = \langle 1^4 \rangle + \langle 1^3 \rangle + \langle 1 \rangle + 2\langle 0 \rangle$$

So

$$[4]'^{\dagger} \otimes \{2\} = [1^4] + 2[21^2] + [31]$$

$$[4]'^{\dagger} \otimes \{1^2\} = [1^4] + [31] + 2[4]$$

while

$$[4]'^{''2} = 2(-[1^4] + [21^2] - [31] + [4])$$

$$[4]'^2 = [21^2] + [4]$$

then we have

$$[4]' \otimes \{2\} = [21^2]$$

$$[4]' \otimes \{1^2\} = [4]$$

$$[4] \subset [4]' \otimes \{1^2\}$$

so

$[4]'$  is symplectic

For  $S_6$ ,  $v = 3$  and  $v \equiv 3 \pmod{4}$  according to (5.43) we have

$$\langle 0 \rangle'^{\dagger} \otimes \{2\} = \langle 1^4 \rangle + 2\langle 1^3 \rangle + \langle 1^2 \rangle + \langle 0 \rangle$$

$$\langle 0 \rangle'^{\dagger} \otimes \{1^2\} = \langle 1^6 \rangle + 2\langle 1^5 \rangle + \langle 1^4 \rangle + \langle 1^2 \rangle + 2\langle 1 \rangle + \langle 0 \rangle$$

so

$$[6]'^{\dagger} \otimes \{2\} = [21^4] + 2[31^3] + [41^2] + [6]$$

$$[6]'^{\dagger} \otimes \{1^2\} = 2[1^6] + 6[21^4] + [41^2] + 2[51] + [6]$$

while

$$[6]'^{\dagger 2} = 2([1^6] - [6] + [51] - [41^2] + [31^4] - [21^4])$$

$$[6]'^2 = [1^6] + [51] + [31^3]$$

then we have

$$[6]' \otimes \{2\} = [31^3]$$

$$[6]' \otimes \{1^2\} = [1^6] + [51]$$

so

$[6]'$  is complex.

For  $S_8$ ,  $v = 4$  and  $v \equiv 0 \pmod{4}$  according to (5.40) we have

$$\langle 0 \rangle'^{\dagger} \otimes \{2\} = \langle 1^8 \rangle + \langle 1^7 \rangle + \langle 1^5 \rangle + 2\langle 1^4 \rangle + \langle 1^3 \rangle + \langle 1 \rangle + 2\langle 0 \rangle$$

$$\langle 0 \rangle'^{\dagger} \otimes \{1^2\} = \langle 1^7 \rangle + 2\langle 1^6 \rangle + \langle 1^5 \rangle + \langle 1^3 \rangle + 2\langle 1^2 \rangle + \langle 1 \rangle$$

so

$$\begin{aligned} [8]'^{\dagger} \otimes \{2\} &= [1^8] + [31^5] + 2[41^4] + [51^3] + [71] \\ &+ 2[8] \end{aligned}$$

$$\begin{aligned} [8]'^{\dagger} \otimes \{1^2\} &= [1^8] + 2[21^6] + [31^5] + [51^3] + 2[61^2] \\ &+ [71] \end{aligned}$$

while

$$\begin{aligned} [8]'' &= 2(-[1^8] + [21^6] - [31^5] + [41^4] - [51^3] + [61^2] \\ &- [71] + [8]) \end{aligned}$$

$$[8]'^2 = [21^6] + [41^4] + [61^2] + [8]$$

then we have

$$[8]' \otimes \{2\} = [41^4] + [8]$$

$$[8]' \otimes \{1^2\} = [21^6] + [61^2]$$

$$[8] \subset [8]' \otimes \{2\}$$

So  $[8]'$  is orthogonal.

For  $S_{10}$ ,  $v = 5$  and  $v \equiv 1 \pmod{4}$  according to (5.41) we have

$$\begin{aligned} \langle 0 \rangle'^{\dagger} \otimes \{2\} &= \langle 1^{10} \rangle + 2\langle 1^9 \rangle + \langle 1^8 \rangle + \langle 1^6 \rangle + 2\langle 1^5 \rangle \\ &+ \langle 1^4 \rangle + \langle 1^2 \rangle + 2\langle 1 \rangle + \langle 0 \rangle \end{aligned}$$

$$\begin{aligned} \langle 0 \rangle'^{\dagger} \otimes \{1^2\} &= \langle 1^8 \rangle + 2\langle 1^7 \rangle + \langle 1^6 \rangle + \langle 1^4 \rangle + 2\langle 1^3 \rangle \\ &+ \langle 1^2 \rangle + \langle 0 \rangle \end{aligned}$$

So

$$\begin{aligned} [10]'^{\dagger} \otimes \{2\} &= 2[1^{10}] + [21^8] + [41^6] + 2[51^5] + [61^4] \\ &+ [81^2] + 2[91] + [10] \end{aligned}$$

$$\begin{aligned} [10]'^{\dagger} \otimes \{1^2\} &= [21^8] + 2[31^7] + [41^6] + [61^4] + 2[71^3] \\ &+ [81^2] + [10] \end{aligned}$$

while

$$\begin{aligned} [10]'' &= 2([1^{10}] - 2[21^8] - [31^7] - [41^6] + [51^5] \\ &- [61^4] + [71^3] - [81^2] + [91] - [10]) \end{aligned}$$

$$[10]'^2 = [1^{10}] + [31^7] + [51^5] + [71^3] + [91]$$

then we have

$$[10]' \otimes \{2\} = [1^{10}] + [51^5] + [91]$$

$$[10]' \otimes \{1^2\} = [31^7] + [71^3]$$

So

$$[10]' \text{ is complex.}$$

Once the plethysms of the basic irrep of  $S_n$  are known, we may evaluate plethysms for any spin irrep of  $S_n$ , since we can always reduce any spin irrep to the product of the basic spin irrep with a sum of ordinary irreps. With the Kronecker square of the basic spin irreps resolved as above, we can now in principle resolve the Kronecker square of any irrep of  $S_n$ .

The classification of the irreps of  $S_n$  as to their complex, orthogonal or symplectic characters follows immediately from the plethysm results just outlined.

For the ordinary irreps of  $S_n$ , all the irreps are real and orthogonal.

For the spin irreps of  $S_n$  we have

If  $(n - k + L)/2$  is odd then  $[\lambda]'$  is complex.

If  $(n - k + 1)/2$  or  $n - k$  are even for  $n = 2v + 1$  we have:

$v \equiv 0, 3 \pmod{4}$  orthogonal

$v \equiv 1, 2 \pmod{4}$  symplectic

while for  $n = 2v$  we have

$v \equiv 0, 1 \pmod{4}$  orthogonal

$v \equiv 2, 3 \pmod{4}$  symplectic

## 5.8 RESOLUTION OF THE BASIC SPIN KRONECKER CUBES FOR $S_{2v+1}$

The evaluation of plethysms  $\langle 0 \rangle'^{\dagger} \otimes \{21\}$  for  $S_{2v+1}$  is equivalent to determining the branching rule

$$O_{2v+1} \downarrow S_{2v+1}$$

$$\Delta \otimes \{21\} \downarrow \langle 0 \rangle'^{\dagger} \otimes \{21\}$$



We know the resolution of  $\Delta \otimes \{21\}$  for  $SO_{2v+1}$ . Because  $\langle 0 \rangle^{\dagger}$  of  $S_{2v+1}$  is self-associated, using the theorem 2 in chapter 1:  $\lambda_i^{\dagger} \times \lambda_j = \lambda_i^{\dagger} \times \tilde{\lambda}_j$ . Then from (4.105) and (4.107,a,b) we can obtain the resolution of the basic spin Kronecker cube of  $S_{2v+1}$  as follows:

$$\langle 0 \rangle^{\dagger} \otimes \{21\} = \langle 0 \rangle^{\dagger} \sum_x (\langle 1^{v-1-3x} \rangle + \langle 1^{v-2-3x} \rangle) \quad (5.48a)$$

$$\begin{aligned} \langle 0 \rangle^{\dagger} \otimes \{3\} = \langle 0 \rangle^{\dagger} \sum_x (\langle 1^{v-12x} \rangle - \langle 1^{v-2-12x} \rangle \\ + \langle 1^{v-3-12x} \rangle + \langle 1^{v-4-12x} \rangle + \langle 1^{v-8-12x} \rangle \\ + \langle 1^{v-9-12x} \rangle - \langle 1^{v-10-12x} \rangle + \langle 1^{v-12-12x} \rangle) \end{aligned} \quad (5.48b)$$

$$\begin{aligned} \langle 0 \rangle^{\dagger} \otimes \{1^3\} = \langle 0 \rangle^{\dagger} \sum_x (\langle 1^{v-2-12x} \rangle + \langle 1^{v-3-12x} \rangle \\ - \langle 1^{v-4-12x} \rangle + 2\langle 1^{v-6-12x} \rangle - \langle 1^{v-8-12x} \rangle \\ + \langle 1^{v-9-12x} \rangle + \langle 1^{v-10-12x} \rangle) \end{aligned} \quad (5.48c)$$

for example we consider  $S_5$  and have

$$\langle 0 \rangle^{\dagger} \otimes \{21\} = \langle 0 \rangle^{\dagger} (\langle 1 \rangle + \langle 0 \rangle)$$

$$= \langle 1 \rangle^{\dagger} + 2\langle 0 \rangle^{\dagger}$$

$$\langle 0 \rangle^{\dagger} \otimes \{3\} = \langle 0 \rangle^{\dagger} (\langle 1^2 \rangle - \langle 0 \rangle)$$

$$= \langle 1 \rangle^{\dagger} + \langle 2 \rangle^{\dagger}$$

$$\langle 0 \rangle^{\dagger} \otimes \{1^3\} = \langle 0 \rangle^{\dagger} \cdot \langle 0 \rangle = \langle 0 \rangle^{\dagger}$$

So for  $S_5$  we have

$$[5]^{\dagger} \otimes \{21\} = [41]^{\dagger} + 2[5]^{\dagger}$$

$$[5]^{\dagger} \otimes \{3\} = [41]^{\dagger} + [32]^{\dagger}$$

$$[5]^{\dagger} \otimes \{1^3\} = [5]^{\dagger}.$$

## CHAPTER 6

### THE REPRESENTATIONS OF ALTERNATING GROUP $A_n$

#### - THE METHOD OF REDUCED NOTATION

#### 6.1 ALTERNATING GROUP $A_n$

The group  $A_n$  is a subgroup of index 2 of  $S_n$  involving only even permutations and is of the order  $n!/2$ .

All classes of  $S_n$  involving only even permutations remain as classes of  $A_n$  with the important exception of those classes for which the even permutations involve only odd cycles of unequal length. In those cases the class splits into two classes of conjugate elements of  $A_n$  each with the same number of elements (Frobenius 1901, Boerner 1970).

The splitting classes of  $A_n$  will be designated as  $(p_1 p_2 \dots p_k)_+$  and  $(p_1 p_2 \dots p_k)_-$  where the  $p_i$  are all odd and

$$p_1 > p_2 > \dots > p_k > 0 \quad (6.1a)$$

and

$$p_1 + p_2 + \dots + p_k = n \quad (6.1b)$$

Thus in  $A_4$  we have the classes  $(1^4)$ ,  $(2^2)$ ,  $(31)_+$ ,  $(31)_-$ .

For spin representation the even  $\alpha$ -regular classes of  $S_n$  remain as  $\alpha$ -regular classes of  $A_n$  though among them there may be splitting classes. In addition there are even classes involving cycles of unequal orders. These latter classes are  $\alpha$ -irregular in  $S_n$  but  $\alpha$ -regular in  $A_n$ . Thus for  $A_6$  we have six  $\alpha$ -regular classes

$$(1^6), (51)_+, (51)_-, (3^2), (31^3), (42) .$$

where the  $\alpha$ -regular (51) class of  $S_6$  has split and  $\alpha$ -irregular (42) class of  $S_2$  is  $\alpha$ -regular in  $A_6$ .

## 6.2 THE IRREPS OF $A_n$

Under  $S_n \downarrow A_n$  the pair of associated irreps of  $S_n$  become equivalent irreps of  $A_n$ , while self-associated irreps of  $S_n$  split into two conjugate irreps of  $A_n$  of the same dimension. As a consequence we shall label the irreps of  $A_n$  by partitions of  $n$ . For the ordinary irreps, if  $(\lambda) \neq (\tilde{\lambda})$  then we use only the partition, while if  $(\lambda) = (\tilde{\lambda})$  we have two conjugate irreps which we shall designate as  $[\lambda]_+$  and  $[\lambda]_-$ . The spin irreps of  $A_n$  are labelled by partitions  $[\lambda]'$  of  $n$  into unequal parts. If  $n-p_\lambda$  is even there are two conjugate spin irreps  $[\lambda]'_+$  and  $[\lambda]'_-$ .

## 6.3 $S_n \downarrow A_n$

In terms of the notation just outlined we may write the  $S_n \downarrow A_n$  branching rules as :

$$[\lambda] \downarrow [\lambda]_+ + [\lambda]_- \quad \text{when } (\lambda) \equiv (\tilde{\lambda}) \quad (6.2a)$$

$$[\lambda], [\tilde{\lambda}] \downarrow [\lambda] \quad \text{when } (\lambda) \neq (\tilde{\lambda}) \quad (6.2b)$$

$$[\lambda]'^+ \downarrow [\lambda]'_+ + [\lambda]'_- \quad \text{when } n-p_\lambda \text{ even} \quad (6.2c)$$

$$[\lambda]', [\tilde{\lambda}]' \downarrow [\lambda]' \quad \text{when } n-p_\lambda \text{ odd} \quad (6.2d)$$

The reduced notation developed for  $S_n$  may be carried over to  $A_n$  to give the  $S_n \downarrow A_n$  branching rules in an essentially  $n$ -independent form as

$$S_n \downarrow A_n$$

$$\langle \mu \rangle \downarrow \langle \mu \rangle_+ + \langle \mu \rangle_- \quad \text{when } \langle \mu \rangle \equiv \langle \tilde{\mu} \rangle \quad (6.3a)$$

$$\langle \mu \rangle, \langle \tilde{\mu} \rangle \downarrow \langle \mu \rangle \quad \text{when } \langle \mu \rangle \neq \langle \tilde{\mu} \rangle \quad (6.3b)$$

$$\langle \mu \rangle' \downarrow \langle \mu \rangle'_+ + \langle \mu \rangle'_- \quad \text{when } n-p_\mu \text{ odd} \quad (6.3c)$$

$$\langle \mu \rangle', \langle \tilde{\mu} \rangle' \downarrow \langle \mu \rangle' \quad \text{when } n-p_\mu \text{ even} \quad (6.3d)$$

In using the above results it is essential to remember that the self-associated irreps are an  $n$ -dependent property.

#### 6.4 THE $A_n \downarrow A_{n-1}$ BRANCHING RULES

Noting the  $S_n \downarrow S_{n-1}$  branching rules given in previous chapter sec. 5.3 together with those for  $S_n \downarrow A_n$  given by (6.3) we find for the ordinary irreps of  $A_n$

$$A_n \downarrow A_{n-1}$$

$$\langle \mu \rangle \downarrow \langle \mu \rangle + \langle \mu/1 \rangle \quad \langle \mu \rangle \neq \langle \tilde{\mu} \rangle \quad (6.4a)$$

$$\langle \mu \rangle_\pm \downarrow \frac{1}{2}(\langle \mu \rangle + \langle \mu/1 \rangle) \quad \langle \mu \rangle = \langle \tilde{\mu} \rangle \quad (6.4b)$$

where the terms on the right  $\langle \rho \rangle$  are to be taken as  $\langle \rho \rangle_+ + \langle \rho \rangle_-$  if  $\langle \rho \rangle = \langle \tilde{\rho} \rangle$ .

Thus we have

$$\langle 1^2 \rangle \downarrow \langle 1^2 \rangle + \langle 1 \rangle$$

leading in  $n = 5$  to

$$[31^2]_{\pm} \downarrow \frac{1}{2}([21^2] + [31]) = [31]$$

and in  $n = 6$  to

$$[41^2] \downarrow [31^2] + [41] = [31^2]_+ + [31^2]_- + [41]$$

Likewise for the spin irreps of  $A_n$  under  $A_n \downarrow A_{n-1}$   
for  $n-p_\mu$  even

$$\langle \mu \rangle' \downarrow (\langle \mu \rangle'^{\dagger} + \langle \mu/1 \rangle'^{\dagger}) = \frac{1}{2} \delta_{\mu_r, 1} \langle \mu_1, \dots, \mu_{r-1} \rangle'^{\dagger} \quad (6.5a)$$

while for  $n-p_\mu$  odd

$$\begin{aligned} \langle \mu \rangle'_{\pm} \downarrow \frac{1}{2}(\langle \mu \rangle'^{\dagger} + \delta_{\mu_r, 1} \langle \mu_1, \dots, \mu_{r-1} \rangle'_{\pm}) \\ - \delta_{\mu_r, 1} \langle \mu_1, \dots, \mu_{r-1} \rangle'_{\mp} \end{aligned} \quad (6.5b)$$

where we put

$$\langle \mu \rangle'^{\dagger} = 2\langle \mu \rangle' \quad \text{if } \langle \mu \rangle \neq \langle \tilde{\mu} \rangle \quad (6.6a)$$

$$= \langle \mu \rangle'_+ + \langle \mu \rangle'_- \quad \text{if } \langle \mu \rangle = \langle \tilde{\mu} \rangle \quad (6.6b)$$

and write  $p_\mu = r$ .

Thus under  $A_n \downarrow A_{n-1}$  we have for  $n-p_\mu$  even

$$\langle 421 \rangle' \downarrow \langle 421 \rangle'^{\dagger} + \langle 321 \rangle'^{\dagger} + \frac{1}{2} \langle 42 \rangle'^{\dagger}$$

and hence for  $A_{13} \downarrow A_{12}$

$$[6421]' \downarrow [5421]_{+}' + [5431]_{-}' + [5321]_{+}' + [6321]_{-}' + [642]'$$

while for  $n-p_\mu$  odd

$$\langle 421 \rangle'_{\pm} \downarrow \frac{1}{2} (\langle 421 \rangle'^{\dagger} + \langle 321 \rangle'^{\dagger} + \langle 42 \rangle'^{\dagger} + \langle 42 \rangle'_{\pm} - \langle 42 \rangle'_{\mp})$$

and hence for  $A_{12} \downarrow A_{11}$

$$[5421]_{\pm}' \downarrow [5321]' + [542]_{\pm}'$$

## 6.5 DIFFERENCE CHARACTERS FOR $A_n$

The simple characters of  $S_n$  which are not self-associated are also simple characters of  $A_n$ . Each self-associated character of  $S_n$  is the sum of two simple characters of  $A_n$ . For the ordinary irreps, the characters of  $A_n$  are found by taking half the value of the character of  $S_n$ , except for the splitting classes  $(p)_{\pm} \equiv (p_1 p_2 \dots p_k)_{\pm}$  where  $p_i = 2\lambda_i - 2i + 1$ . In this case the character of the splitting classes in  $A_n$  are given by (Frobenius 1901)

$$\chi_{(p)_{\pm}}^{[\lambda]} = \frac{1}{2} [(-1)^{(n-k)/2} \pm i^{(n-k)/2} (p_1 p_2 \dots p_k)^{\frac{1}{2}}] \quad (6.7a)$$

and

$$\chi_{(p)_{\pm}}^{[\lambda]_{-}} = \frac{1}{2} [ (-1)^{(n-k)/2} \mp i^{(n-k)/2} (p_1 p_2 \cdots p_k)^{\frac{1}{2}} ] \quad (6.7b)$$

If the difference character is defined as

$$\chi_{(\rho)}^{[\lambda]''} = \chi_{(\rho)}^{[\lambda]_{+}} - \chi_{(\rho)}^{[\lambda]_{-}} \quad (6.8)$$

then

$$\chi_{(\rho)_{\pm}}^{[\lambda]''} = \pm i^{(n-k)/2} (p_1 p_2 \cdots p_k)^{\frac{1}{2}} \quad (6.9)$$

and vanishes for all other classes.

In the case of spin irreps of  $S_n$  those irreps labelled by partitions  $(\lambda)$  into  $k$  unequal odd parts, with  $n-k$  even, are self-associated and split into two conjugate irreps  $[\lambda]_{\pm}'$  of  $A_n$ . Here we find for the splitting classes  $(\lambda)_{\pm}$

$$\chi_{(\lambda)_{\pm}}^{[\lambda]_{\pm}'} = \frac{1}{2} [ (-1)^{(k-n \bmod 2)/2} \pm i^{(n-k)/2} (\lambda_1 \lambda_2 \cdots \lambda_k)^{\frac{1}{2}} ] \quad (6.10a)$$

and

$$\chi_{(\lambda)_{\pm}}^{[\lambda]_{\mp}'} = \frac{1}{2} [ (-1)^{(k-n \bmod 2)/2} \mp i^{(n-k)/2} (\lambda_1 \lambda_2 \cdots \lambda_k)^{\frac{1}{2}} ] \quad (6.10b)$$

and hence

$$\chi_{(\lambda)_{\pm}}^{[\lambda]''} = \pm i^{(n-k)/2} (\lambda_1 \lambda_2 \cdots \lambda_k)^{\frac{1}{2}} \quad (6.11)$$

while for the class  $(\lambda)$  where all parts of  $(\lambda)$  are unequal and one or more even



$$\chi_{(\lambda)}^{[\lambda]'} = \pm \frac{1}{2} i^{(n-k)/2} (\lambda_1 \lambda_2 \cdots \lambda_k)^{\frac{1}{2}} \quad (6.12)$$

and hence

$$\chi_{(\lambda)}^{(\lambda)''} = \pm i^{(n-k)/2} (\lambda_1 \lambda_2 \cdots \lambda_k)^{\frac{1}{2}} \quad (6.13)$$

In all other classes the spin character for the conjugate irreps  $[\lambda]_{\pm}'$  of  $A_n$  are simply half of their corresponding values in  $S_n$ , and the difference character vanish.

It follows that the associated characters of  $S_n$  decompose into real characters of  $A_n$ , while the self-associated just are the characters of  $A_n$  and hence real, while the self-associated characters of  $S_n$ , decompose in a pair of real conjugate characters of  $A_n$  if  $n - k \equiv 0 \pmod{4}$ . Otherwise we obtain a pair of complex characters of  $A_n$ .

By way of an example of the notation developed herein we have given the spin character table of  $A_8$  in Table XII.

## 6.6 KRONECKER PRODUCTS OF THE IRREPS OF $A_n$

The analysis of Kronecker products of irreps of  $A_n$  given here draws heavily upon the methods developed in Chapter 5 but with a rather more extensive use of the properties of difference characters. Our results are summarized in seven algorithms which cover all possible cases. Most of the evaluations are made by first resolving a Kronecker product in  $S_n$  and then using the  $S_n \downarrow A_n$  branching rules together with the properties of the

TABLE XII: Table of spin characters for the  $\alpha$ -regular classes of  $A_8$ .

Class	$(1^8)$	$(31^5)$	$(51^3)$	$(3^21^2)$	$(71)_+$	$(71)_-$	$(62)$	$(53)_+$	$(53)_-$
Order	1	112	1344	1120	2880	2880	3360	1360	1344
$[8]'$	8	4	2	2	1	1	0	1	1
$[71]_+'$	24	6	1	0	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$	0	-1	-1
$[71]_-'$	24	6	1	0	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$	0	-1	-1
$[62]_+'$	56	4	-1	-1	0	0	$-i\sqrt{3}$	1	1
$[62]_-'$	56	4	-1	-1	0	0	$+i\sqrt{3}$	1	1
$[53]_+'$	56	-2	-1	2	0	0	0	$\frac{-1+i\sqrt{15}}{2}$	$\frac{-1-i\sqrt{15}}{2}$
$[53]_-'$	56	-2	-1	2	0	0	0	$\frac{-1-i\sqrt{15}}{2}$	$\frac{-1+i\sqrt{15}}{2}$
$[521]'$	64	-4	1	-2	1	1	0	-1	-1
$[431]'$	48	-6	2	0	-1	-1	0	1	1

difference characters in assigning the  $A_n$  irreps to the appropriate  $A_n$  product.

The first algorithm covers all cases that do not involve members of a conjugate pair of  $A_n$  irreps.

#### Algorithm I

(1) To resolve  $\langle \mu \rangle \langle \nu \rangle$ ,  $\langle \mu \rangle \langle \nu \rangle'$  or  $\langle \mu \rangle' \langle \nu \rangle'$  with

$$\langle \mu \rangle \langle \nu \rangle = \langle \mu \rangle \langle \nu \rangle \quad (6.14a)$$

$$\langle \mu \rangle \langle \nu \rangle' = \langle \mu \rangle \langle \nu \rangle'^{\dagger} / 2 \quad (6.14b)$$

$$\langle \mu \rangle' \langle \nu \rangle' = \langle \mu \rangle'^{\dagger} \langle \nu \rangle'^{\dagger} / 4 \quad (6.14c)$$

(2) In each case the right-hand-side involves  $S_n$  Kronecker product. Resolve these using the algorithms given in chapter 5.

(3) Restrict the resulting  $S_n$  irreps to those of  $A_n$  using (6.3a) to (6.3d).

The second algorithm resolves the products  $\langle \mu \rangle \langle \nu \rangle_{\pm}$  for ordinary irreps of  $A_n$ .

#### Algorithm II

Evaluate  $\langle \mu \rangle \langle \nu \rangle$  for  $S_n$  giving

$$\langle \mu \rangle \langle \nu \rangle = g_{\mu\nu}^{\rho} \langle \rho \rangle$$

and restrict the right-hand-side to  $A_n$  using (6.3a) and (6.3b).

(2) Divide the coefficients associated with every term found in (1) by 2. The integral part of the resulting coefficients is the number of times its corresponding irrep occurs in  $\langle \mu \rangle \langle \nu \rangle_+$  and in  $\langle \mu \rangle \langle \nu \rangle_-$ . If there is no residue the resolution is complete.

(3) The only possible residue will be a term

$$\langle \nu \rangle^\dagger = \langle \nu \rangle_+ + \langle \nu \rangle_-$$

If the character  $\chi_{(p)}^{\langle \mu \rangle} = +1$  then  $\langle \nu \rangle_+$  is assigned to  $\langle \mu \rangle \langle \nu \rangle_+$  and  $\langle \nu \rangle_-$  to  $\langle \mu \rangle \langle \nu \rangle_-$  while if  $\chi_{\langle p \rangle}^{\langle \mu \rangle} = -1$  the opposite assignment is made.

The third algorithm treats the case  $\langle \mu \rangle_\pm \langle \nu \rangle'$ .

### Algorithm III

(1) Evaluate  $\langle \mu \rangle \langle \nu \rangle'^\dagger$  for  $S_n$  using algorithm II of chapter 5 to give

$$\langle \mu \rangle \langle \nu \rangle'^\dagger = g_{\mu\nu}^{\rho} \langle \rho \rangle'^\dagger$$

and restrict the above results to  $A_n$  using (6.3c) and (6.3d) to give

$$(\langle \mu \rangle_+ + \langle \mu \rangle_-) \langle \nu \rangle' = \frac{1}{2} g_{\mu\nu}^{\rho} \langle \rho \rangle'^\dagger$$

(2) Divide the coefficients associated with every term found in (1) by 2. The integral part of the resulting coefficients is the number of times its corresponding irrep

occurs in  $\langle \mu \rangle_+ \langle \nu \rangle'$  and  $\langle \mu \rangle_- \langle \nu \rangle'$ . If there is no residue the resolution is complete.

(3) The only possible residue will be a term

$$\langle p \rangle'^{\dagger} = \langle p \rangle'_+ + \langle p \rangle'_-$$

If the character  $\chi_{(p)}^{\langle \nu \rangle'} = +1$  then  $\langle p \rangle'_+$  is assigned to  $\langle \mu \rangle_+ \langle \nu \rangle'$  and  $\langle p \rangle'_-$  to  $\langle \mu \rangle_- \langle \nu \rangle'$  while if  $\chi_{(p)}^{\langle \nu \rangle'} = -1$  the opposite assignment is made.

The fourth algorithm covers the case  $\langle \mu \rangle \langle \nu \rangle'_\pm$ .

#### Algorithm IV

(1) Evaluate  $\langle \mu \rangle \langle \nu \rangle'^{\dagger}$  for  $S_n$  using algorithm II of chapter 5 to give

$$\langle \mu \rangle \langle \nu \rangle'^{\dagger} = g_{\mu\nu}^{\rho} \langle \rho \rangle'^{\dagger}$$

and restrict the right-hand-side to  $A_n$  using (6.3c) and (6.3d).

(2) Divide the coefficients associated with every term found in (1) by 2. The integral part of the resulting coefficients is the number of times its corresponding irrep occurs in  $\langle \mu \rangle \langle \nu \rangle'_+$  and in  $\langle \mu \rangle \langle \nu \rangle'_-$ . If there is no residue the resolution is complete.

(3) The only possible residue will be a term

$$\langle v \rangle^{\dagger} = \langle v \rangle_{+}^{\dagger} + \langle v \rangle_{-}^{\dagger}$$

If the character  $\chi_{(v)}^{\langle \mu \rangle} = 1$  then  $\langle v \rangle_{+}^{\dagger}$  is assigned to  $\langle \mu \rangle \langle v \rangle_{+}^{\dagger}$  and  $\langle v \rangle_{-}^{\dagger}$  to  $\langle \mu \rangle \langle v \rangle_{-}^{\dagger}$  while if  $\chi_{(v)}^{\langle \mu \rangle} = -1$  the opposite assignment is made.

The next three algorithms require the use of difference characters. Let

$$\langle \mu \rangle^{\dagger} = \langle \mu \rangle_{+} + \langle \mu \rangle_{-} \quad (6.15a)$$

and

$$\langle \mu \rangle^{\prime\prime} = \langle \mu \rangle_{+} - \langle \mu \rangle_{-} \quad (6.15b)$$

then generally

$$\begin{aligned} \langle \mu \rangle_{\pm} \langle v \rangle_{\pm} = \frac{1}{4} [ & \langle \mu \rangle^{\dagger} \langle v \rangle^{\dagger} \pm \langle \mu \rangle^{\dagger} \langle v \rangle^{\prime\prime} \pm \langle \mu \rangle^{\prime\prime} \langle v \rangle^{\dagger} \\ & + \langle \mu \rangle^{\prime\prime} \langle v \rangle^{\prime\prime} ] \end{aligned} \quad (6.16a)$$

and

$$\begin{aligned} \langle \mu \rangle_{\pm} \langle v \rangle_{\mp} = \frac{1}{4} [ & \langle \mu \rangle^{\dagger} \langle v \rangle^{\dagger} \pm \langle \mu \rangle^{\dagger} \langle v \rangle^{\prime\prime} \mp \langle \mu \rangle^{\prime\prime} \langle v \rangle^{\dagger} \\ & - \langle \mu \rangle^{\prime\prime} \langle v \rangle^{\prime\prime} ] \end{aligned} \quad (6.16b)$$

For ordinary irreps of  $A_n$

$$\langle \mu \rangle^{\prime\prime} \langle v \rangle = \langle \mu \rangle \langle v \rangle^{\prime\prime} = \langle \mu \rangle^{\prime\prime} \langle v \rangle^{\prime\prime} = 0 \quad \text{if } (\mu) = (v) \quad (6.17a)$$

and hence

$$\langle \mu \rangle_{\pm} \langle \nu \rangle_{\pm} = \langle \mu \rangle_{\pm} \langle \nu \rangle_{\mp} = \frac{1}{4} \langle \mu \rangle \langle \nu \rangle \quad (6.17b)$$

while if  $(\mu) \equiv (\nu)$  then

$$\langle \mu \rangle_{\pm}^2 = \frac{1}{4} [\langle \mu \rangle^{\dagger 2} \pm 2 \langle \mu \rangle^{\dagger} \langle \mu \rangle'' + \langle \mu \rangle''^2] \quad (6.18a)$$

and

$$\langle \mu \rangle_{+} \langle \mu \rangle_{-} = \frac{1}{4} [\langle \mu \rangle^{\dagger 2} - \langle \mu \rangle''^2] \quad (6.18b)$$

For the spin irreps of  $A_n$  there are two distinct cases:

(a) spin irreps labelled by partition into unequal parts with one or more even, and

(b) spin irreps labelled by partitions into unequal odd parts.

In case (a) we find

$$\langle \nu \rangle^{\dagger} \langle \nu \rangle'' = 0 \quad (6.19)$$

and hence

$$\langle \nu \rangle_{\pm}^2 = \frac{1}{4} [\langle \nu \rangle^{\dagger 2} + \langle \nu \rangle''^2] \quad (6.20a)$$

and

$$\langle \nu \rangle_{+} \langle \nu \rangle_{-} = \frac{1}{4} [\langle \nu \rangle^{\dagger 2} - \langle \nu \rangle''^2] \quad (6.20b)$$

while in case (b) we find

$$\langle \nu \rangle^{\dagger} \langle \nu \rangle'' \neq 0 \quad (6.21)$$

and hence

$$\langle v \rangle_{\pm}^{\dagger 2} = \frac{1}{4} [\langle v \rangle_{\pm}^{\dagger 2} \pm 2 \langle v \rangle_{\pm}^{\dagger} \langle v \rangle_{\pm}'' + \langle v \rangle_{\pm}''^2] \quad (6.22a)$$

and

$$\langle v \rangle_{+}^{\dagger} \langle v \rangle_{-}^{\dagger} = \frac{1}{4} [\langle v \rangle_{+}^{\dagger 2} - \langle v \rangle_{-}''^2] \quad (6.22b)$$

For  $(\mu) \neq (v)$  we obtain

$$\langle \mu \rangle_{\pm}^{\dagger} \langle \mu \rangle_{\pm}'' = \langle \mu \rangle_{\pm}'' \langle v \rangle_{\pm}^{\dagger} = \langle \mu \rangle_{\pm}'' \langle v \rangle_{\pm}'' = 0 \quad (6.23)$$

and hence

$$\langle \mu \rangle_{\pm}^{\dagger} \langle v \rangle_{\pm}^{\dagger} = \langle \mu \rangle_{\pm}^{\dagger} \langle v \rangle_{\mp}^{\dagger} = \frac{1}{4} \langle \mu \rangle_{\pm}^{\dagger} \langle v \rangle_{\pm}^{\dagger} \quad (6.24)$$

We can now state the remaining three algorithms. The first deals with the cases  $\langle \mu \rangle_{\pm} \langle \mu \rangle_{\pm}$  and  $\langle \mu \rangle_{\pm} \langle v \rangle_{\mp}$  for ordinary irreps of  $A_n$ .

Algorithm V

(1) Evaluate  $\langle \mu \rangle \langle v \rangle$  for  $S_n$  to give

$$\langle \mu \rangle \langle v \rangle = g_{\mu v}^{\rho} \langle \rho \rangle$$

and restrict the right-hand-side to  $A_n$  using (6.3a) and (6.3b).

(2) If  $\langle \mu \rangle \neq \langle v \rangle$  then

$$\langle \mu \rangle_{\pm} \langle v \rangle_{\pm} = \langle \mu \rangle_{\pm} \langle v \rangle_{\mp} = \frac{1}{4} \langle \mu \rangle \langle v \rangle$$



(3) If  $\langle \mu \rangle = \langle \nu \rangle$  then evaluate  $\langle \mu \rangle''^2$  for  $S_n$ .

$$\langle \mu \rangle''^2 = g_{\mu\mu}^{\rho} \langle \rho \rangle$$

where

$$g_{\mu\mu}^{\rho} = i^{n-k} \chi_{(p)}^{\langle \rho \rangle}$$

These results are then used in (6.18a) and (6.18b) together with the  $S_n \downarrow A_n$  branching rules noting that

$$\langle \mu \rangle \langle \mu \rangle'' = \pm (\langle \mu \rangle_+ - \langle \mu \rangle_-) \text{ as } \chi_{(p)}^{\langle \mu \rangle} = \pm 1.$$

The penultimate algorithm covers the cases  $\langle \mu \rangle_{\pm}' \langle \mu \rangle_{\pm}'$  and  $\langle \mu \rangle_{\pm}' \langle \nu \rangle_{\pm}'$ .

Algorithm VI

(1) Evaluate  $\langle \mu \rangle, {}^{\dagger} \langle \nu \rangle, {}^{\dagger}$  for  $S_n$  using the algorithm IV of chapter 5 to give

$$\langle \mu \rangle, {}^{\dagger} \langle \nu \rangle, {}^{\dagger} = g_{\mu\nu}^{\rho} \langle \rho \rangle^{\dagger}$$

and restrict the right-hand-side to  $A_n$  using (6.3a) and (6.3b).

(2) If  $(\mu) \neq (\nu)$  then use (6.24).

(3) If  $(\mu) = (\nu)$  and  $(\mu)$  has unequal parts with one or more even then evaluate  $\langle \mu \rangle''^2$  for  $S_n$  noting that

$$\langle \mu \rangle, {}''^2 = g_{\mu\mu}^{\rho} \langle \rho \rangle$$

and restricting the right-hand-side to  $A_n$  using (6.3a) and (6.3b). These results are then used in (6.22a) and (6.22b) to complete the evaluation.

(4) If  $(\mu) = (v)$  and  $(\mu)$  has unequal odd parts evaluate  $\langle \mu \rangle'^{n^2}$  for  $S_n$  to give

$$\langle \mu \rangle'^{n^2} = g_{\mu\mu}^{\rho} \langle \rho \rangle$$

where

$$g_{\mu\mu}^{\rho} = i^{n-k} \chi_{(\mu)}^{\langle \rho \rangle}$$

and restrict to the right-hand-side to  $A_n$  using (6.3a) and (6.3b). Furthermore

$$\langle \mu \rangle'^{\dagger} \langle \mu \rangle' = \pm (\langle \mu \rangle_+ - \langle \mu \rangle_-) \text{ as } \chi_{(\mu)}^{\langle \mu \rangle'^{\dagger}} = \pm 1$$

The final algorithm concerns the cases  $\langle \mu \rangle_{\pm} \langle v \rangle_{\pm}'$  and  $\langle \mu \rangle_{\pm} \langle v \rangle_{\mp}'$ .

#### Algorithm VII

(1) Evaluate  $\langle \mu \rangle \langle v \rangle'^{\dagger}$  for  $S_n$  using algorithm III of chapter 5 to give

$$\langle \mu \rangle \langle v \rangle'^{\dagger} = g_{\mu v}^{\rho} \langle \rho \rangle'^{\dagger}$$

and restrict the right-hand side to  $A_n$  using (6.3c) and (6.3d).

(2) If  $\langle v \rangle^{\dagger}$  has unequal parts with one or more even then

$$\langle \mu \rangle_{\pm} \langle v \rangle_{\pm}^{\dagger} = \langle \mu \rangle_{\pm} \langle v \rangle_{\mp}^{\dagger} = \frac{1}{4} (g_{\mu\nu}^{\rho} \langle \rho \rangle^{\dagger})$$

(3) If  $\langle v \rangle^{\dagger}$  has unequal odd parts then evaluate

$$\langle \mu \rangle^{\dagger} \langle v \rangle^{\dagger} = g_{\mu\nu}^{\rho} \langle \rho \rangle^{\dagger}$$

where

$$g_{\mu\nu}^{\rho} = i^{n-k} \chi_{(\nu)}^{\langle \rho \rangle^{\dagger}}$$

and restrict to  $A_n$ . If

$$\chi_{(\nu)}^{\langle v \rangle^{\dagger}} = \chi_{(\nu)}^{\langle \mu \rangle^{\dagger}}$$

then

$$\langle \mu \rangle^{\dagger} \langle v \rangle^{\dagger} = \langle \mu \rangle^{\dagger} \langle v \rangle^{\dagger} = (-1)^{\frac{n-k}{2}} (\langle v \rangle_{+}^{\dagger} - \langle v \rangle_{-}^{\dagger})$$

while if

$$\chi_{(\nu)}^{\langle v \rangle^{\dagger}} = - \chi_{(\nu)}^{\langle \mu \rangle^{\dagger}}$$

$$\text{then } \langle \mu \rangle \langle v \rangle^{\dagger} = - \langle \mu \rangle^{\dagger} \langle v \rangle^{\dagger} = (-1)^{\frac{n-k}{2}} (\langle v \rangle_{+}^{\dagger} - \langle v \rangle_{-}^{\dagger}).$$

The evaluation is completed by noting (6.16a) and (6.16b).

The above algorithms are essentially free of the need to use character tables. The only ordinary character required are of an especially simple form and may be readily evaluated, if required, using the Littlewood theorem given in chapter 1.

The spin character  $\chi_{(p)}^{<\mu>}$  may be evaluated using the formulas of Morris also given in chapter 1.

The application of the above algorithms is best seen in the following examples for  $A_6$ . First consider the  $A_6$  product  $[51]_+^+ [42]_-^+$  using algorithm VI. Specializing to  $S_6$  we readily find that

$$\begin{aligned} [51]_+^+ [42]_-^+ &= 2[51]^+ + 4[42]^+ + 4[41^2]^+ + 2[3^2]^+ \\ &\quad + 8[321]^+ \end{aligned}$$

Using of (6.3a) and (6.3b) for  $S_6 \downarrow A_6$  gives

$$\begin{aligned} [51]_+^+ [42]_-^+ &= 4[51] + 8[42] + 8[41^2] + 4[3^2] \\ &\quad + 8[321]_+ + 8[321]_- \end{aligned}$$

Finally use of (6.24) results in

$$\begin{aligned} [51]_+^+ [42]_-^+ &= [51] + 2[42] + 2[41^2] + [3^2] + 2[321]_+ \\ &\quad + 2[321]_- \end{aligned}$$

Now consider the  $A_6$  product  $[51]_+^+ [51]_-^+$  again using algorithm VI. Using the results of chapter 5 we find

$$\begin{aligned} [51]_+^+ [51]_-^+ &= [6]^+ + 2[51]^+ + 3[42]^+ + 4[41^2]^+ + 2[3^2]^+ \\ &\quad + 5[321]. \end{aligned}$$

Under  $S_6 \downarrow A_6$  the right-hand-side becomes

$$= 2[6] + 4[51] + 6[42] + 8[41^2] + 4[3^2] \\ + 5[321]_+ + 5[321]_-$$

Furthermore

$$[51]'''[51]'' = [6] - [42] + [321] - [2^2 1^2] + [1^6] \\ = 2[6] - 2[42] + [321]_+ + [321]_-$$

and

$$[51]'^{\dagger}[51]'' = -[321]_+ + [321]_-$$

Use of (6.22b) then yields

$$[51]_+^{\dagger}[51]_-^{\dagger} = [51] + 2[42] + 2[41^2] + [3^2] + [321]_+ \\ + [321]_-$$

while use of (6.22a) yields

$$[51]_+^{\dagger}[51]_+^{\dagger} = [6] + [51] + [42] + 2[41^2] + [3^2] \\ + [321]_+ + 2[321]_-$$

and

$$[51]_-^{\dagger}[51]_-^{\dagger} = [6] + [51] + [42] + 2[41^2] + [3^2] + 2[321]_+ \\ + [321]_-$$

## 6.7 SYMMETRIZED KRONECKER SQUARES AND THE CLASSIFICATION OF IRREPS OF $A_n$

We now consider the resolving of the Kronecker squares of ordinary and spin irreps of  $A_n$  into their symmetric and antisymmetric terms.

First we consider the ordinary irreps of  $A_n$  that are not members of a conjugate pair. Since in this case  $\langle \mu \rangle \downarrow \langle \mu \rangle$  under  $S_n \downarrow A_n$  we can simply evaluate  $\langle \mu \rangle \otimes \{2\}$  and  $\langle \mu \rangle \otimes \{1^2\}$  as for  $S_n$  using in chapter 5 and then use (6.3a) and (6.3b) to make the  $S_n \downarrow A_n$  reductions.

For example since in  $S_n$  we have (cf. Table X)

$$\langle 2 \rangle \otimes \{1^2\} = \langle 1^2 \rangle + \langle 1^3 \rangle + \langle 21 \rangle + \langle 31 \rangle$$

we have for  $S_7$

$$[52] \otimes \{1^2\} = [51^2] + [41^3] + [421] + [3^21]$$

and hence for  $A_7$

$$[52] \otimes \{1^2\} = [51^2] + [41^3]_+ + [41^3]_- + [421] + [3^21]$$

For the conjugate pairs  $\langle \mu \rangle_{\pm}$  of  $A_n$  it is necessary to use difference characters and to note that

$$\begin{aligned} \langle \mu \rangle_{\pm} \otimes \{2\} &= \frac{1}{2} [\langle \mu \rangle^+ \otimes \{2\} + \langle \mu \rangle^- \otimes \{2\} + \langle \mu \rangle^+ \langle \mu \rangle^- \\ &\quad - \langle \mu \rangle_{\pm}^2] \end{aligned} \tag{6.25a}$$

and

$$\begin{aligned} \langle \mu \rangle_{\pm} \otimes \{1^2\} &= \frac{1}{2} [\langle \mu \rangle^{\dagger} \otimes \{1^2\} + \langle \mu \rangle^{\prime\prime} \otimes \{1^2\} + \langle \mu \rangle^{\dagger} \langle \mu \rangle^{\prime\prime} \\ &\quad - \langle \mu \rangle_{\pm}^2] \end{aligned} \quad (6.25b)$$

The ordinary  $S_n$  plethysms can be evaluated as in chapter 5 and the products  $\langle \mu \rangle^{\dagger} \langle \mu \rangle^{\prime\prime}$  and  $\langle \mu \rangle_{\pm}^2$  as in the previous section. To evaluate  $\langle \mu \rangle^{\prime\prime} \otimes \{2\}$  and  $\langle \mu \rangle^{\prime\prime} \otimes \{1^2\}$  we note that if for  $A_n$

$$\langle \mu \rangle^{\prime\prime 2} = 2g_{\mu\mu}^{\rho} \langle \rho \rangle + \langle \mu \rangle_{+} + \langle \mu \rangle_{-}$$

then

$$\langle \mu \rangle^{\prime\prime} \otimes \{2\} = g_{\mu\mu}^{\rho} \langle \rho \rangle + \langle \mu \rangle_{\mp} \quad \text{if } \frac{1}{2}(n-k) = \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \quad (6.26a)$$

and

$$\langle \mu \rangle^{\prime\prime} \otimes \{1^2\} = g_{\mu\mu}^{\rho} \langle \rho \rangle + \langle \mu \rangle_{\pm} \quad \text{if } \frac{1}{2}(n-k) = \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \quad (6.26b)$$

Use of the above results readily leads to

$$[321]_{\pm} \otimes \{1^2\} = 2[41^2] + [321]_{\pm}$$

for  $A_6$ .

For the spin irreps of  $A_n$  we need to treat the two cases  $n = 2v$  and  $n = 2v+1$  separately. The primary need is to evaluate the plethysms for the basic spin irrep  $\langle 0 \rangle^{\dagger}$  since any other plethysm involving spin irreps can be reduced to a plethysm involving the basic spin irreps and those involving ordinary irreps.

In the case of  $A_{2v}$  the squares of the basic spin irreps are resolved by use of (5.46a) and (5.46b) of chapter 5

followed by use of the  $S_n \downarrow A_n$  branching rules. If  $v \equiv 0, 1 \pmod{4}$  the basic spin irrep of  $A_{2v}$  is orthogonal while if  $v \equiv 2, 3 \pmod{4}$  it is symplectic.

In the case of  $A_{2v+1}$  difference characters for the basic spin irreps are used exactly as in (6.25a) and (6.25b) leading to the conclusion that if  $v \equiv 1, 3 \pmod{4}$   $\langle 0 \rangle_{\pm}'$  are complex while if  $v \equiv 0 \pmod{4}$   $\langle 0 \rangle_{\pm}'$  are orthogonal and if  $v \equiv 2 \pmod{4}$   $\langle 0 \rangle_{\pm}'$  are symplectic.

Thus the irreps of  $A_n$  may be classified as following.

For the ordinary irreps of  $A_n$  we have:

If  $[\tilde{\lambda}] = [\lambda]$  in  $S_n$  and  $n-k \not\equiv 0 \pmod{4}$  then  $[\lambda]_{\pm}'$  of  $A_n$  is complex, where  $k$  is the number of  $p_i = 2\lambda_i - 2i + 1$  with  $p_i > 0$ , while all other ordinary irreps of  $A_n$  are real and orthogonal.

For the spin irreps of  $A_n$  we have:

If  $(n-k)/2$  is odd then  $[\lambda_1 \lambda_2 \cdots \lambda_k]_{\pm}'$  are complex, while if  $(n-k-1)$  or  $(n-k)/2$  is even then for  $n = 2v+1$  we have

$v \equiv 0, 3 \pmod{4}$  orthogonal

$v \equiv 1, 2 \pmod{4}$  symplectic

and for  $n = 2v$  we have

$v \equiv 0, 1 \pmod{4}$  orthogonal

$v \equiv 2, 3 \pmod{4}$  symplectic.



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APPENDIX  
PUBLICATIONS

A large part of the results of this thesis have been published in co-authorship with B.G. Wybourne and R.C. King on three papers listed as follows:

The symmetric group: branching rules, products and plethysms for spin representation, 1981. J. Phys:A 14 327

The Alternating group: branching rules, products and plethysms for ordinary and spin representations. 1981. J. Phys. A 14 1835.

Symmetrized powers of rotation group representations. 1981, J. Phys. A. 14 2509.